

A SPACE-TIME CHARACTERIZATION OF THE KERR-NEWMAN METRIC

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ABSTRACT. In the present paper, the characterization of the Kerr metric found by Marc Mars is extended to the Kerr-Newman family. A simultaneous alignment of the Maxwell field, the Ernst two-form of the pseudo-stationary Killing vector field, and the Weyl curvature of the metric is shown to imply that the space-time is locally isometric to domains in the Kerr-Newman metric. The paper also presents an extension of Ionescu and Klainerman's null tetrad formalism to explicitly include Ricci curvature terms.

1. INTRODUCTION

There are relatively few known exact solutions, which have metrics that can be easily written down in closed form, to the Einstein equations in the asymptotically flat case. Among the most well-known of such solutions are the Kerr family [Ker63] of axially-symmetric, stationary vacuum space-times, which represent the exterior space-time of a spinning massive object, and the Kerr-Newman family [NCC⁺65] of axially-symmetric, stationary, electrovac space-times, which represent the exterior space-time of a spinning, electrically charged, massive object. A natural question to ask about special solutions such as these is whether they are stable or unique, where stability or uniqueness is chosen among some suitable class. While much progress had been made toward the uniqueness problem, less can be said about the stability problem.

It should not be surprising that the first results of this kind came in the context of the special static, spherically symmetric members of the Kerr and Kerr-Newman families: the Schwarzschild and Reissner-Nordström solutions respectively. It has been known since the 1920's [JR05] that the Schwarzschild family completely parametrizes the spherically symmetric solutions to Einstein's vacuum equations; a similar result is later obtained for the Reissner-Nordström family for spherically symmetric solutions of the electrovac equations. These results now go under the name of Birkhoff's theorem. In particular, Birkhoff's theorem essentially states that spherical symmetry implies staticity and asymptotic flatness of the space-time. The next step forward came in the 1960's, when Werner Israel established [Isr67, Isr68] what is, loosely speaking, the converse of Birkhoff's theorem: a static, asymptotically flat space-time that is regular on the event horizon must be spherically symmetric. Brandon Carter's 1973 Les Houches report [Car73] finally sparked an attempt to similarly characterize the Kerr and Kerr-Newman families: he showed that asymptotically flat, stationary, and axially-symmetric solutions to the vacuum

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(electrovac) equations form a two-parameter (three-) family. Between D. C. Robinson [Rob75], P. O. Mazur [Maz82], and G. L. Bunting [Bun83], Carter's program was completed and the Kerr and Kerr-Newman families are established as essentially the unique solutions to the asymptotically flat, stationary, axially-symmetric Einstein's equations.

A different approach was taken by Walter Simon [Sim84a, Sim84b] to study the characterization of the Kerr and Kerr-Newman families among stationary solutions. He constructed three-index tensors that are, heuristically speaking, complexified versions of the Cotton tensors on the stationary spatial slices (to be more precise, the manifold of trajectories generated by the time-like Killing vector field). By considering the multiple moments of a stationary, asymptotically flat end, Simon showed that the vanishing of the three-index tensor is equivalent to the multiple moments being equal to those of the Kerr and Kerr-Newman families. Simon's work was later extended by Marc Mars [Mar99] to the construction of the so-called Mars-Simon tensor, which is a four-index tensor constructed relative to space-time quantities, as opposed to Simon's original construction relative to the induced metric on the spatial slices. As was shown by Mars, the vanishing of the Mars-Simon tensor indicates an alignment of the principal null directions of the Ernst two-form (for definition see Section 2) with those of the Weyl curvature tensor, with the particular proportionality factor allowing one to write down the local form of the metric explicitly and verify that the space-time is locally isometric to the Kerr space-time.

The method employed by Marc Mars and the present paper bears much similarity to the work of R. Debever, N. Kamran, and R. G. McLenaghan [DKM84], in which the authors assumed (i) the space-time is of Petrov type D , (ii) the principal null directions of the Maxwell tensor align (nonsingularly) with that of the Weyl tensor, (iii) a technical hypothesis to allow the use of the generalized Goldberg-Sachs theorem (see Chapter 7 in [SKM⁺02] for example and references), and integrated the Newman-Penrose variables to arrive at explicit local forms of the metric in terms of several constants that can be freely specified. In view of the work of Debever et al., the assumptions taken in this paper merely guarantees that their hypotheses (i) and (ii) hold, and that (iii) becomes ancillary to a stronger condition derived herein that circumvents the Goldberg-Sachs theorem as well as prescribes definite values for all but three (mass, angular momentum, and charge) of the free constants.

In the current work, we extend the construction of Mars to define a four-tensor analogous to the Mars-Simon tensor *and*, in addition, a two-form such that their simultaneous vanishing guarantees the simultaneous alignment of the principal null directions of the Ernst two-form, the Maxwell field, and the Weyl tensor, with proportionality factors that allow us to write down the local form of the metric and demonstrate a local isometry to Kerr-Newman space-time. It is worth mentioning the work of Donato Bini et al. [BCJM04] in which they keep the same definition of the Mars-Simon tensor, while modifying the definition of the Simon three-tensor with a source term that corresponds to the stress-energy tensor associated to the electromagnetic field. They were then able to show that the vanishing of the modified Simon tensor implies also the alignment of principal null directions. In the present work, we absorb the source term into the Mars-Simon tensor itself using only space-time quantities by sacrificing a need for an auxiliary two-form, thus we are able to argue in much of the same way as Mars [Mar99] an explicit computation

for the metric expressed in local coördinates, thereby giving a characterization of the Kerr-Newman space-time.

In a forthcoming paper we hope to use this characterization, combined with the Carleman estimate techniques of Ionescu and Klainerman [IK07a, IK07b] to obtain an analogous uniqueness result for smooth stationary charged black holes.

We should note that the characterization found here is essentially local, analogous to Theorem 1 in [Mar00] (see Theorem 2 below). The global feature of the space-time, namely asymptotic flatness, is only used to a priori prescribe the values of certain constants using the mass and charge at infinity (compare Corollary 3 below). The present paper is organized as follows: In Section 2, we first review the concept of complex anti-self-dual two-forms and their properties, then we present the basic assumptions on the space-time under consideration, followed by a quick review of Killing vector fields, and conclude with the principal definitions and a statement of the main theorem and corollary. In Section 3, we demonstrate the technique of the proof for the main theorem through explicit construction of a local isometry, using the tools of the null tetrad formalism of Ionescu and Klainerman [IK07a]. In Section 4 we prove the corollary. We also include an appendix extending the framework established by Ionescu and Klainerman to explicitly include terms coming from Ricci curvature (terms which were not necessary in [IK07a, IK07b] since they consider vacuum Einstein metrics), and including a dictionary between the coefficients in this formalism and those of the Newman-Penrose system.

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2. SET-UP AND DEFINITIONS

2.1. Complex anti-self-dual two-forms. On a four dimensional Lorentzian space-time (\mathcal{M}, g_{ab}) , the Hodge-star operator $*$: $\Lambda^2 T^* \mathcal{M} \rightarrow \Lambda^2 T^* \mathcal{M}$ is a linear transformation on the space of two-forms. In index notation,

$$*X_{ab} = \frac{1}{2} \epsilon_{abcd} X^{cd}$$

where ϵ_{abcd} is the volume form and index-raising is done relative to the metric g . Since we take the metric signature to be $(-, +, +, +)$, we have that $** = -Id$, which introduces a complex structure on the space $\Lambda^2 T^* \mathcal{M}$. By complexifying and extending the action of $*$ by linearity, we can split $\Lambda^2 T^* \mathcal{M} \otimes_{\mathbb{R}} \mathbb{C}$ into the eigenspaces Λ_{\pm} of $*$ with eigenvalues $\pm i$. We say that an element of $\Lambda^2 T^* \mathcal{M} \otimes_{\mathbb{R}} \mathbb{C}$ is *self-dual* if it is an eigenvector of $*$ with eigenvalue i , and we say that it is *anti-self-dual* if it has eigenvalue $-i$. It is easy to check that given a real-valued two-form X_{ab} , the two form

$$(1) \quad \mathcal{X}_{ab} := \frac{1}{2}(X_{ab} + i^* X_{ab})$$

is anti-self-dual, while its complex conjugate $\bar{\mathcal{X}}_{ab}$ is self-dual.

In the sequel we shall, in general, write elements of $\Lambda^2 T^* \mathcal{M}$ with upper-case Roman letters, and their corresponding anti-self-dual forms with upper-case calligraphic letters. The projection

$$X_{ab} = \mathcal{X}_{ab} + \bar{\mathcal{X}}_{ab}$$

is a natural consequence of (1).

Here we record some product properties [Mar99] of two-forms:

$$(2a) \quad X_{ac} Y_b^c - {}^* X_{ac} {}^* Y_b^c = \frac{1}{2} g_{ab} X_{cd} Y^{cd}$$

$$(2b) \quad X_{ac} {}^* X_b^c = \frac{1}{4} g_{ab} X_{cd} {}^* X^{cd}$$

$$(2c) \quad \mathcal{X}_{ac} \mathcal{Y}_b^c + \mathcal{Y}_{ac} \mathcal{X}_b^c = \frac{1}{2} g_{ab} \mathcal{X}_{cd} \mathcal{Y}^{cd}$$

$$(2d) \quad \mathcal{X}_{ac} \mathcal{X}_b^c = \frac{1}{4} g_{ab} \mathcal{X}_{cd} \mathcal{X}^{cd}$$

$$(2e) \quad \mathcal{X}_{ac} X_b^c - \mathcal{X}_{bc} X_a^c = 0$$

$$(2f) \quad \mathcal{X}_{ab} Y^{ab} = \mathcal{X}_{ab} \mathcal{Y}^{ab}$$

$$(2g) \quad \mathcal{X}_{ab} \bar{\mathcal{Y}}^{ab} = 0$$

Now, the projection operator $\mathcal{P}_\pm : \Lambda^2 T^* \mathcal{M} \otimes_{\mathbb{R}} \mathbb{C} \rightarrow \Lambda_\pm$ can be given in index notation as

$$(\mathcal{P}_+ X)_{ab} = \bar{\mathcal{I}}_{abcd} X^{cd}$$

$$(\mathcal{P}_- X)_{ab} = \mathcal{I}_{abcd} X^{cd}$$

$$\text{where } \mathcal{I}_{abcd} = \frac{1}{4} (g_{ac} g_{bd} - g_{ad} g_{bc} + i \epsilon_{abcd})$$

With the complex tensor \mathcal{I}_{abcd} , we can define with the notation

$$(3) \quad (\mathcal{X} \tilde{\otimes} \mathcal{Y})_{abcd} := \frac{1}{2} \mathcal{X}_{ab} \mathcal{Y}_{cd} + \frac{1}{2} \mathcal{Y}_{ab} \mathcal{X}_{cd} - \frac{1}{3} \mathcal{I}_{abcd} \mathcal{X}_{ef} \mathcal{Y}^{ef}$$

a symmetric bilinear product taking two anti-self-dual forms to a complex $(0, 4)$ -tensor. It is simple to verify that such a tensor automatically satisfies the algebraic symmetries of the Weyl conformal tensor: i) it is antisymmetric in its first two, and last two, indices $(\mathcal{X} \tilde{\otimes} \mathcal{Y})_{abcd} = -(\mathcal{X} \tilde{\otimes} \mathcal{Y})_{bacd} = -(\mathcal{X} \tilde{\otimes} \mathcal{Y})_{abdc}$ ii) it is symmetric swapping the first two and the last two sets of indices $(\mathcal{X} \tilde{\otimes} \mathcal{Y})_{abcd} = (\mathcal{X} \tilde{\otimes} \mathcal{Y})_{cdab}$ iii) it verifies the first Bianchi identity $(\mathcal{X} \tilde{\otimes} \mathcal{Y})_{abcd} + (\mathcal{X} \tilde{\otimes} \mathcal{Y})_{bcad} + (\mathcal{X} \tilde{\otimes} \mathcal{Y})_{cabd} = 0$ and iv) it is trace-free $(\mathcal{X} \tilde{\otimes} \mathcal{Y})_{abcd} g^{ac} = 0$. For lack of a better name, this product will be referred to as a *symmetric spinor product*, using the fact that if we represent in spinor coördinates $\mathcal{X}_{ab} = f_{AB} \epsilon_{A'B'}$ and $\mathcal{Y}_{ab} = h_{AB} \epsilon_{A'B'}$ (where $f_{AB} = f_{BA}$, and similarly for h_{AB}), the product can be written as

$$(\mathcal{X} \tilde{\otimes} \mathcal{Y})_{abcd} = f_{(AB} h_{CD)} \epsilon_{A'B'} \epsilon_{C'D'}$$

where (\cdot) denotes complete symmetrization of the indices.

2.2. The basic assumptions on the space-time and some notational definitions. We consider a space-time (\mathcal{M}, g_{ab}) and a Maxwell two-form H_{ab} on \mathcal{M} satisfying the following basic assumptions

- (A1) \mathcal{M} is a four-dimensional, orientable, paracompact, simply-connected manifold.

- (A2) g_{ab} is a smooth Lorentzian metric on \mathcal{M} . H_{ab} is a smooth two-form.
- (A3) The metric g_{ab} and the Maxwell form H_{ab} satisfy the Einstein-Maxwell field equations. Namely

$$\begin{aligned} R_{ab} &= T_{ab} \\ \nabla_{[c} H_{ab]} &= 0 \\ \nabla^a H_{ab} &= 0 \end{aligned}$$

where $T_{ab} = 2H_{ac}H_b{}^c - \frac{1}{2}g_{ab}H_{cd}H^{cd} = 4\mathcal{H}_{ac}\bar{\mathcal{H}}_b{}^c$ is the rescaled stress-energy tensor, which is traceless and divergence free by construction, and square brackets $[\cdot]$ around indices means full anti-symmetrization.

- (A4) (\mathcal{M}, g_{ab}) admits a non-trivial smooth Killing vector field t^a , and the Maxwell field H_{ab} inherits the Killing symmetry, i.e. its Lie derivative $\mathcal{L}_t H_{ab} = 0$.

In the sequel we will state a local and a global version of the result. For the local theorem, we need to assume

- (L) the Killing vector field t^a is time like somewhere on the space-time (\mathcal{M}, g_{ab}) , and H_{ab} is non-null on \mathcal{M} . (In other words, writing the anti-self-dual part $\mathcal{H}_{ab} = \frac{1}{2}(H_{ab} + i^*H_{ab})$, we require $\mathcal{H}_{ab}\mathcal{H}^{ab} \neq 0$ everywhere on \mathcal{M} .)

And for the global result, we assume

- (G) that (\mathcal{M}, g_{ab}) contains a stationary asymptotically flat end \mathcal{M}^∞ where t^a tends to a time translation at infinity, with the Komar mass M of t^a non-zero in \mathcal{M}^∞ . We also assume the total charge $q = \sqrt{q_E^2 + q_B^2}$ of the Maxwell field, where q_E and q_B denote the electric and magnetic charges, is non-zero in \mathcal{M}^∞ .

Remark 1. We quickly recall the definition of stationary asymptotically flat end: \mathcal{M}^∞ is an open submanifold of \mathcal{M} diffeomorphic to $(t_0, t_1) \times (\mathbb{R}^3 \setminus \bar{B}(R))$ with the metric stationary in the t variable, $\partial_t g_{ab} = 0$, and satisfying the decay condition

$$|g_{ab} - \eta_{ab}| + |r\partial g_{ab}| \leq Cr^{-1}$$

for some constant C ; r is the radial coördinate on \mathbb{R}^3 and η is the Minkowski metric. In addition, we will also require a decay condition for the Maxwell field

$$|H_{ab}| + |r\partial H_{ab}| \leq C'r^{-2}$$

for some constant C' .

We record here some notational definitions: R_{abcd} is the Riemann curvature tensor, with the standard decomposition

$$R_{abcd} = W_{abcd} + \frac{1}{2}(R_{ac}g_{bd} + R_{bd}g_{ac} - R_{ad}g_{bc} - R_{bc}g_{ad}) - \frac{1}{6}R(g_{ac}g_{bd} - g_{ad}g_{bc})$$

where W_{abcd} is the conformal (Weyl) curvature tensor, $R_{ac} = R_{abcd}g^{bd}$ the Ricci curvature tensor, and R the scalar curvature. For the electro-vac system, this reduces to

$$R_{abcd} = W_{abcd} + \frac{1}{2}(T_{ac}g_{bd} + T_{bd}g_{ac} - T_{ad}g_{bc} - T_{bc}g_{ad})$$

For a $(0, 4)$ tensor K_{abcd} satisfying the algebraic symmetries of the Riemann tensor, we define the left- and right-duals

$$\begin{aligned} {}^*K_{abcd} &= \frac{1}{2}\epsilon_{abef}K^{ef}{}_{cd} \\ K^*{}_{abcd} &= \frac{1}{2}K_{ab}{}^{ef}\epsilon_{efcd} \end{aligned}$$

In general the left- and right-duals are *not* equal. If, in addition, K_{abcd} is also trace-free (i.e., is a Weyl field in the sense defined in [CK93]), a simple calculation shows that the left- and right-duals are equal. Therefore we can define the anti-self-dual complex Weyl curvature tensor

$$\mathcal{C}_{abcd} = \frac{1}{2}(W_{abcd} + i^*W_{abcd})$$

It may be of independent interest to note that in the electro-vac case

$$\mathcal{C}_{abcd} = (\mathcal{P}_- R \mathcal{P}_-)_{abcd} = \mathcal{I}_{abef} R^{efgh} \mathcal{I}_{ghcd}$$

(when the scalar curvature $R \neq 0$, it also presents a contribution to this projection). In other words, treating the Riemann curvature tensor as a map from $\Lambda^2 T^* \mathcal{M} \otimes_{\mathbb{R}} \mathbb{C}$ to itself, the Weyl curvature takes $\Lambda_+ \rightarrow \Lambda_+$ and $\Lambda_- \rightarrow \Lambda_-$, whereas the Kulkarni-Nomizu product of Ricci curvature with the metric induces an intertwining map that takes $\Lambda_- \rightarrow \Lambda_+$ and vice versa.

Lastly, we define the following notational shorthand for Lorentzian “norms” of tensor fields. For an arbitrary (j, k) -tensor $Z_{b_1 b_2 \dots b_k}^{a_1 a_2 \dots a_j}$, we write

$$Z^2 = g_{a_1 a'_1} g_{a_2 a'_2} \dots g_{a_j a'_j} g^{b_1 b'_1} \dots g^{b_k b'_k} Z_{b_1 b_2 \dots b_k}^{a_1 a_2 \dots a_j} Z_{b'_1 b'_2 \dots b'_k}^{a'_1 a'_2 \dots a'_j}$$

for the inner-product of Z with itself. Note that in the semi-Riemannian setting, Z^2 can take arbitrary sign.

2.3. The Killing symmetry. Given (\mathcal{M}, g_{ab}) a smooth, four-dimensional Lorentzian manifold, and assuming that it admits a smooth Killing vector field t^a , we can define the Ernst two-form

$$(4) \quad F_{ab} = \nabla_a t_b - \nabla_b t_a = 2\nabla_a t_b$$

the second equality a consequence of the Killing equation. As is well-known, the Ernst two-form satisfy

$$(5) \quad \nabla_c F_{ab} = 2\nabla_c \nabla_a t_b = 2R_{dcab} t^d$$

This directly implies a divergence-curl system (in other words, a Maxwell equation with source terms) satisfied by the two-form

$$\begin{aligned} \nabla_{[c} F_{ab]} &= 0 \\ \nabla^a F_{ab} &= -2R_{db} t^d \end{aligned}$$

Here we encounter one of our primary differences from [Mar99]: a space-time satisfying the Einstein vacuum equations is Ricci-flat, and the above implies that the Ernst two-form satisfies the sourceless Maxwell equations. In particular, for the vacuum case, we have

$$\nabla_{[c} \mathcal{F}_{ab]} = 0$$

and a calculation then verifies that

$$\nabla_{[c} (\mathcal{F}_a)_{b]} t^b = 0.$$

Thus from simple-connectivity, an Ernst potential σ is constructed for

$$\nabla_a \sigma = \mathcal{F}_{ab} t^b .$$

In the non-vacuum case that this paper deals with, this construction cannot be exactly carried through. However, the essence of the construction above is the following fact disjoint from the semi-Riemannian structure of our setup: consider a smooth manifold \mathcal{M} , a smooth differential form X , and a smooth vector-field v . We have the defining relation

$$\mathcal{L}_v X = i_v \circ dX + d \circ i_v X$$

where \mathcal{L}_v stands for the Lie derivative relative to the vector-field v , and i_v is the interior derivative. Thus if X is a closed form, and v is a symmetry of X (i.e. $\mathcal{L}_v X = 0$), we must have $i_v X$ is closed also.

Applying to the Einstein-Maxwell equations, we take X to be the anti-self-dual Maxwell form \mathcal{H}_{ab} , which by Maxwell's equations is closed. The vector-field v is naturally the Killing field t^a , so we conclude that the complex-valued one-form $\mathcal{H}_{ab} t^a$ is closed, and since \mathcal{M} is taken to be simply connected, also exact. In the sequel we will use the complex-valued function Ξ , which is defined by

$$(6) \quad \nabla_b \Xi = \mathcal{H}_{ab} t^a .$$

Notice that *a priori* Ξ is only defined up to the addition of a constant. In the global case (making the assumption (G)), we can use the asymptotic decay of the Maxwell field to require that $\Xi \rightarrow 0$ at spatial infinity and fix Ξ uniquely. The function Ξ takes the place of the Ernst potential σ used in [Mar99].

Lastly, we record here two calculations used in the sequel: first we write down explicitly the derivative of \mathcal{F}_{ab}

$$(7) \quad \begin{aligned} \nabla_c \mathcal{F}_{ab} &= (R_{dcab} + i R_{dcab}^*) t^d \\ &= 2\mathcal{C}_{dcab} t^d + \frac{1}{2}(T_{ad} g_{bc} + T_{bc} g_{ad} - T_{ac} g_{bd} - T_{bd} g_{ac}) t^d \\ &\quad + \frac{i}{2}(T_d^e \epsilon_{ecab} + T_c^f \epsilon_{dfab}) t^d \end{aligned}$$

we will also need the following fact about Killing vector fields. Consider the product $*F_{ab} *F_{cd} = \frac{1}{4} \epsilon_{abef} \epsilon_{cdgh} F^{ef} F^{gh}$. We can expand the product of the Levi-Civita symbol/volume form in terms of the metric:

$$\epsilon_{ijkl} \epsilon^{qrst} = -24 g_i^{[q} g_j^r g_k^s g_l^{t]}$$

By explicit computation using this expansion, we arrive at the fact

$$\begin{aligned} *F_{mx} t^x *F_{ny} t^y &= \frac{1}{2} F_{ab} F^{ab} (t_m t_n - t_x t^x g_{mn}) + g_{mn} F_{xa} t^x F^{ya} t_y - F_{nx} t^x F_{my} t^y \\ &\quad + F^{bx} t_x t_m F_{nb} + F^{bx} t_x t_n F_{mb} + t_x t^x F_{ma} F_n^a \end{aligned}$$

Writing $t^2 = t_a t^a$, we use the fact $\nabla_b t^2 = t^a F_{ba}$ and obtain equation (13) from [Mar99]:

$$(8) \quad \begin{aligned} *F_{mx} t^x *F_{ny} t^y &= \frac{1}{2} F_{ab} F^{ab} (t_m t_n - g_{mn} t^2) + g_{mn} \nabla_a t^2 \nabla^a t^2 - \nabla_m t^2 \nabla_n t^2 \\ &\quad + t_m F_{nb} \nabla^b t^2 + t_n F_{mb} \nabla^b t^2 + t^2 F_{ma} F_n^a \end{aligned}$$

2.4. The Mars-Simon tensor for Kerr-Newman space-time; statement of the main theorems. We first state the main result of this paper, which establishes a purely local characterization of the Kerr-Newman metric. This formulation is comparable to that of Theorem 1 in [Mar00]. The conditions given below on the constants C_2 and C_4 are analogous to the conditions for the constants l and c in the aforementioned theorem.

Theorem 2 (Main Local Theorem). *Assuming (A1)-(A4) and (L), and assuming that there exists a complex scalar P , a normalization for Ξ , and a complex constant C_1 such that*

- (1) $P^{-4} = -C_1^2 \mathcal{H}_{ab} \mathcal{H}^{ab}$
- (2) $\mathcal{F}_{ab} = 4\Xi \mathcal{H}_{ab}$
- (3) $\mathcal{C}_{abcd} = 3P(\mathcal{F} \otimes \mathcal{H})_{abcd}$

then we can conclude

- (1) *there exists a complex constant C_2 such that $P^{-1} - 2\Xi = C_2$;*
- (2) *there exists a real constant C_4 such that $t_a t^a + 4|\Xi|^2 = C_4$.*

If C_2 further satisfies that $C_1 \bar{C}_2$ is real, and that C_4 is such that $|C_2|^2 - C_4 = 1$, then we also have

- (3) $\mathfrak{A} = |C_1|^2 P \bar{P} (\Im C_1 \nabla P)^2 + (\Im C_1 P)^2$ *is a positive real constant on the manifold¹,*
- (4) *and (\mathcal{M}, g_{ab}) is locally isometric to a Kerr-Newman space-time of total charge $|C_1|$, angular momentum $\sqrt{\mathfrak{A}} C_1 \bar{C}_2$, and mass $C_1 \bar{C}_2$.*

The local theorem yields, via a simple argument, the following characterization of the Kerr-Newman metric among stationary asymptotically flat solutions to the Einstein-Maxwell system.

Corollary 3 (Main Global Result). *We assume (A1)-(A4) and (G), and let q_E , q_B , and M be the electric charge, magnetic charge, and Komar mass of the space-time at one asymptotic end. We choose the normalization for Ξ such that it vanishes at spatial infinity. If we assume there exists a complex function P defined wherever $\mathcal{H}^2 \neq 0$ such that*

- (1) $P^{-4} = -(q_E + iq_B)^2 \mathcal{H}_{ab} \mathcal{H}^{ab}$ *when $\mathcal{H}^2 \neq 0$*
- (2) $\mathcal{F}_{ab} = (4\Xi - \frac{2M}{q_E + iq_B}) \mathcal{H}_{ab}$ *everywhere*
- (3) $\mathcal{C}_{abcd} = 3P(\mathcal{F} \otimes \mathcal{H})_{abcd}$ *when P is defined*

then we can conclude that

- (1) \mathcal{H}^2 *is non-vanishing globally,*
- (2) $\mathfrak{A} = (q_E^2 + q_B^2) P \bar{P} (\Im(q_E + iq_B) \nabla P)^2 + (\Im(q_E + iq_B) P)^2$ *is a real-valued positive constant on the manifold,*
- (3) *and (\mathcal{M}, g_{ab}) is everywhere locally isometric to a Kerr-Newman space-time of total charge $q = \sqrt{q_E^2 + q_B^2}$, angular momentum $\sqrt{\mathfrak{A}} M$, and mass M .*

¹ \Im will be used to denote the imaginary part of an expression. Notice that \mathfrak{A} is well defined even though C_1 can be replaced by $-C_1$. One should observe the freedom to replace C_1 by $-C_1$ also manifests in the remainder of this paper; it shall not be further remarked upon.

For ease of notation, we write the complex scalar P , the complex anti-self-dual form \mathcal{B}_{ab} , and the complex anti-self-dual Weyl field \mathcal{Q}_{abcd} for the following expressions

$$(9a) \quad P^4 := -\frac{1}{C_1^2 \mathcal{H}_{ab} \mathcal{H}^{ab}}$$

$$(9b) \quad \mathcal{B}_{ab} := \mathcal{F}_{ab} + (2\bar{C}_3 - 4\bar{\Xi})\mathcal{H}_{ab}$$

$$(9c) \quad \mathcal{Q}_{abcd} := \mathcal{C}_{abcd} - 3P(\mathcal{F} \tilde{\otimes} \mathcal{H})_{abcd}$$

By an abuse of language, in the sequel, the statement “ $\mathcal{B}_{ab} = 0$ ” will be understood to mean the alignment condition (2) in Theorem 2 when we work under assumption (L), or the alignment condition (2) in Corollary 3 when we work under assumption (G), with suitably defined constants and normalizations. Similarly, the statement “ $\mathcal{Q}_{abcd} = 0$ ” will be taken to mean the existence of a suitable function P such that the appropriate alignment condition (3) is satisfied under suitable conditions.

We end this section with a heuristic motivation of why the pair $\mathcal{B}_{ab}, \mathcal{Q}_{abcd}$ is a generalization of the Mars-Simon tensor constructed in [IK07a]. Assuming (G) and suppose we have \mathcal{B}_{ab} and \mathcal{Q}_{ab} both vanishing, and we take the $q \rightarrow 0$ Kerr limit. Formally we define the quantity

$$\mathcal{G}_{ab} = -\frac{2M}{q_E + iq_B} \mathcal{H}_{ab}$$

when $q \neq 0$. The vanishing of \mathcal{B}_{ab} becomes

$$\mathcal{G}_{ab} = \mathcal{F}_{ab} / (1 - \frac{2(q_E + iq_B)}{M} \bar{\Xi})$$

and P satisfies

$$P^4 = -\frac{4M^2}{(q_E + iq_B)^4 \mathcal{G}_{ab} \mathcal{G}^{ab}}$$

Then we have

$$0 = \mathcal{Q}_{abcd} = \mathcal{C}_{abcd} + \frac{3}{2M(-\frac{1}{4M^2} \mathcal{G}_{kl} \mathcal{G}^{kl})^{1/4}} (\mathcal{F} \tilde{\otimes} \mathcal{G})_{abcd}$$

Now, formally taking $q \rightarrow 0$, we have that $\mathcal{B}_{ab} = 0 \rightarrow \mathcal{G}_{ab} = \mathcal{F}_{ab}$, and

$$\mathcal{Q}_{abcd} = 0 \rightarrow \mathcal{C}_{abcd} = -\frac{3}{(-4M^2 \mathcal{F}_{kl} \mathcal{F}^{kl})^{1/4}} (\mathcal{F} \tilde{\otimes} \mathcal{F})_{abcd}$$

which by inspection is the same vanishing condition imposed by the Mars-Simon tensor in [IK07a] or the vanishing condition in Lemma 5 of [Mar99] (the difference of a factor of 2 is due to a factor of 2 difference in the definitions of anti-self-dual two-forms and of the Ernst two-form).

3. PROOF OF THE MAIN LOCAL THEOREM

Throughout this section we assume the statements (A1)-(A4) and (L). The arguments in this section, except for Lemma 4 and Proposition 5, closely mirrors the arguments given in [Mar99], with several technical changes to allow the application to electrovac space-times. Using the precise statement of Theorem 2, C_3 should be taken to be 0 in this section. We keep the notation C_3 to make explicit the applicability of the computations in the global case.

We start first with some consequences of assumption (L)

Lemma 4. *If \mathcal{B}_{ab} vanishes identically on \mathcal{M} , then we have that*

- (1) $\mathcal{F}_{ab}\mathcal{F}^{ab}$ only vanishes on sets of co-dimension ≥ 1 ,
- (2) $\mathcal{F}_{ab}\mathcal{F}^{ab} = 0 \implies \mathcal{F}_{ab} = 0$,
- (3) The Killing vector field t^a is non-null on a dense subset of \mathcal{M} .

Proof. Squaring the alignment condition implied by the vanishing of \mathcal{B}_{ab} gives

$$\mathcal{F}^2 = (4\bar{\Xi} - 2\bar{C}_3)^2 \mathcal{H}^2.$$

By assumption (L), if the left-hand side vanishes, then $4\bar{\Xi} - 2\bar{C}_3 = 0$, and using the alignment condition again, we have $\mathcal{F}_{ab} = 0$. This proves claim (2).

Suppose \mathcal{F}_{ab} vanishes on some small open set δ , then necessarily $\nabla_a t_b = 0$ on δ . Furthermore, we have that $\bar{\Xi}$ must be locally constant as shown above, and thus $\nabla_a \bar{\Xi} = \mathcal{H}_{ba} t^b = 0$. But

$$\nabla_a \bar{\Xi} \nabla^a \bar{\Xi} = \mathcal{H}_{ba} \mathcal{H}^{ca} t_c t^b = \frac{1}{4} \mathcal{H}_{ab} \mathcal{H}^{ab} t_c t^c = 0$$

and since the Maxwell field is non-null, we have that t^a must be a parallel null vector in δ . If t^a is not the zero vector, however, we must have t^a being an eigenvector, and hence a principal null direction, of \mathcal{H}_{ab} , with eigenvalue zero: this contradicts the fact that \mathcal{H}_{ab} is non-null. If $t^a = 0$ on a small neighborhood δ , however, t^a must vanish everywhere on \mathcal{M} since it is Killing, contradicting assumption (A4). This proves assertion (1).

Lastly, assume that $t^2 = 0$ on some small open set δ , which implies $\nabla_a t^2 = 0$ and $\square_g t^2 = 0$ on the neighborhood. Using (8), we deduce

$${}^*F_{mx} t^x {}^*F_{ny} t^y = \frac{1}{2} F^2 t_m t_n$$

Taking the trace in m, n , we have

$${}^*F_{mx} {}^*F^{my} t_y t^x = 0$$

Using the fact that

$$F_{mx} F^{my} t^x t_y = \nabla_m t^2 \nabla^m t^2 = 0, \quad \mathcal{F}_{ac} \bar{\mathcal{F}}_b{}^c = \frac{1}{4} (F_{ac} F_b{}^c + {}^*F_{ac} {}^*F_b{}^c)$$

we have

$$\mathcal{F}_{ac} \bar{\mathcal{F}}_b{}^c t^a t^b = 0$$

Now, since $\mathcal{B}_{ab} = 0$, this implies that

$$|2C_3 - 4\bar{\Xi}|^2 T_{ab} t^a t^b = 0$$

on the open set δ . If the first factor is identically zero in an open subset $\delta' \subset \delta$, then $\bar{\Xi}$ is locally constant and arguing the same way as above we get a contradiction. Therefore we can assume, without loss of generality, that $T_{ab} t^a t^b = 0$. Now consider the identity

$$0 = \square_g t^2 = \nabla^b (t^a F_{ba}) = \frac{1}{2} F^{ba} F_{ba} - 2R_{ab} t^a t^b$$

The last term vanishes by the assumption, and implies that $F^{ba} F_{ba} = 0$; thus ${}^*F_{mx} t^x = 0$. Therefore

$$\nabla_a t^2 = t^b F_{ab} = 2t^b \mathcal{F}_{ab}$$

in δ , and hence

$$0 = \square_g t^2 = \mathcal{F}_{ab} \mathcal{F}^{ab} - 2R_{ab} t^a t^b$$

and so $\mathcal{F}_{ab} \mathcal{F}^{ab} = 0$ identically on δ , which we have just shown is impossible. Assertion (3) then follows. \square

We can then prove claim (1) in Theorem 2:

Proposition 5. *If \mathcal{B}_{ab} and \mathcal{Q}_{abcd} both vanish on \mathcal{M} , then $P^{-1} - 2\Xi$ is constant.*

Proof. We start by calculating $\mathcal{H}^{ab}\nabla_c\mathcal{B}_{ab}$. Using (7),

$$\begin{aligned}\mathcal{H}^{ab}\nabla_c\mathcal{F}_{ab} &= 2[\mathcal{Q}_{dcab} + 3P(\mathcal{F}\tilde{\otimes}\mathcal{H})_{dcab}]t^d\mathcal{H}^{ab} \\ &\quad + \frac{1}{2}(T_{ad}\mathcal{H}^a{}_c + T_{bc}\mathcal{H}_d{}^b - T_{ac}\mathcal{H}^a{}_d - T_{bd}\mathcal{H}_c{}^b)t^d \\ &\quad + i(T_d{}^{e*}\mathcal{H}_{ec} + T_c{}^{f*}\mathcal{H}_{df})t^d \\ &= 2[\mathcal{Q}_{dcab} + 3P(\mathcal{F}\tilde{\otimes}\mathcal{H})_{dcab}]t^d\mathcal{H}^{ab} + 2(T_{ad}\mathcal{H}^a{}_c + T_{bc}\mathcal{H}_d{}^b)t^d \\ &= 2\mathcal{Q}_{dcab}\mathcal{H}^{ab}t^d + P(3\mathcal{F}_{dc}\mathcal{H}_{ab}\mathcal{H}^{ab} + \mathcal{H}_{dc}\mathcal{F}_{ab}\mathcal{H}^{ab})t^d \\ &\quad + 8(\mathcal{H}_{af}\bar{\mathcal{H}}_d{}^f\mathcal{H}^a{}_c + \mathcal{H}_{bf}\bar{\mathcal{H}}_c{}^f\mathcal{H}_d{}^b)t^d \\ &= 2\mathcal{Q}_{dcab}\mathcal{H}^{ab}t^d + P(3[\mathcal{B}_{dc} - (2\bar{C}_3 - 4\bar{\Xi})\mathcal{H}_{dc}]\mathcal{H}_{ab}\mathcal{H}^{ab} \\ &\quad + \mathcal{H}_{dc}[\mathcal{B}_{ab} - (2\bar{C}_3 - 4\bar{\Xi})\mathcal{H}_{ab}]\mathcal{H}^{ab})t^d + 4\mathcal{H}_{ab}\mathcal{H}^{ab}\bar{\mathcal{H}}_{dc}t^d\end{aligned}$$

where we used (2d) and (9b) in the last equality. Using (9a), we simplify to

$$\begin{aligned}\mathcal{H}^{ab}\nabla_c\mathcal{F}_{ab} &= 2\mathcal{Q}_{dcab}\mathcal{H}^{ab}t^d - \frac{3}{C_1^2P^3}\mathcal{B}_{dc}t^d + \frac{4}{C_1^2P^3}(2\bar{C}_3 - 4\bar{\Xi})\mathcal{H}_{dc}t^d \\ &\quad + \mathcal{H}_{dc}\mathcal{B}_{ab}\mathcal{H}^{ab}t^d - \frac{4}{C_1^2P^4}\bar{\mathcal{H}}_{dc}t^d\end{aligned}$$

Applying the condition $\mathcal{Q}_{abcd} = 0$ and $\mathcal{B}_{ab} = 0$ and (6), we have

$$\mathcal{H}^{ab}\nabla_c\mathcal{F}_{ab} = \frac{4}{C_1^2P^3}(2\bar{C}_3 - 4\bar{\Xi})\nabla_c\Xi - \frac{4}{C_1^2P^4}\nabla_c\bar{\Xi}$$

On the other hand, we can calculate

$$\mathcal{H}^{ab}\nabla_c[(2\bar{C}_3 - 4\bar{\Xi})\mathcal{H}_{ab}] = -4\mathcal{H}^{ab}\mathcal{H}_{ab}\nabla_c\bar{\Xi} + \frac{1}{2}(2\bar{C}_3 - 4\bar{\Xi})\nabla_c(\mathcal{H}_{ab}\mathcal{H}^{ab})$$

So putting them altogether we have

$$0 = \mathcal{H}^{ab}\nabla_c\mathcal{B}_{ab} = \frac{4}{C_1^2P^3}(\bar{C}_3 - 2\bar{\Xi})(2\nabla_c\Xi - \nabla_c\frac{1}{P})$$

By the arguments used in the proof of Lemma 4, Ξ is not locally constant and so $C_3 \neq 2\Xi$ densely. The above expression (and continuity) then shows that $2\Xi - \frac{1}{P}$ is constant. \square

In what follows I'll write $C_2 = P^{-1} - 2\Xi + C_3$.

Remark 6. *In the global case (where we assume (G) instead of (L)), the decay condition given by asymptotic flatness shows that 2Ξ and $1/P$ both vanish at spatial infinity, and so $C_2 = C_3 = M/(q_E - iq_B)$ everywhere.*

The next proposition demonstrates assertion (2) in Theorem 2.

Proposition 7. *Assuming the vanishing of \mathcal{B}_{ab} and \mathcal{Q}_{abcd} , we have the following identities*

$$(10a) \quad t^2 = -\left|\frac{1}{P} - C_2\right|^2 + C_4$$

$$(10b) \quad (\nabla P)^2 = -\frac{t^2}{C_1^2}$$

$$(10c) \quad C_1 \square_g P = -\frac{2}{C_1 \bar{C}_1 P \bar{P}} (\bar{C}_1 C_2 - (|C_2|^2 - C_4) \bar{C}_1 \bar{P})$$

where C_4 is a real-valued constant.

Proof. We can calculate

$$\nabla_a t^2 = 2t^b \nabla_a t_b = -F_{ba} t^b = -2\Re[\mathcal{F}_{ba} t^b]$$

The vanishing of \mathcal{B}_{ab} and Proposition 5 together imply

$$\nabla_a t^2 = -4\Re[(2\bar{\Xi} - \bar{C}_3)\mathcal{H}_{ba} t^b] = -2\Re[(\frac{1}{P} - \bar{C}_2)\nabla_a \frac{1}{P}] = -\nabla_a \left|\frac{1}{P} - C_2\right|^2$$

The first claim follows as \mathcal{M} is simply connected. Next, from Proposition 5 we get

$$\nabla_a P = \nabla_a \frac{1}{2\Xi + C_2 - C_3} = -\frac{2\nabla_a \Xi}{(2\Xi + C_2 - C_3)^2} = -2P^2 \mathcal{H}_{ba} t^b$$

So

$$\nabla_a P \nabla^a P = 4P^4 \mathcal{H}_{ba} t^b \mathcal{H}^{ca} t_c = P^4 \mathcal{H}^2 t^2 = -\frac{t^2}{C_1^2}$$

where we used (2d) and the definition for P . We can also calculate directly the D'Alembertian

$$\begin{aligned} \square_g P &= -2\nabla^a (P^2 \mathcal{H}_{ba} t^b) \\ &= -2\mathcal{H}_{ba} (2P \nabla^a P t^b + \frac{1}{2} P^2 F^{ab}) \\ &= 2\mathcal{H}_{ba} (4P^3 \mathcal{H}^{ca} t_c t^b + \frac{1}{2} P^2 \mathcal{F}^{ba}) \\ &= 2P^3 \mathcal{H}^2 t^2 + 2P^2 (\frac{1}{P} - \bar{C}_2) \mathcal{H}^2 \\ &= 2P^2 \mathcal{H}^2 \left[P \left(-\left|\frac{1}{P} - C_2\right|^2 + C_4 \right) + \frac{1}{P} - \bar{C}_2 \right] \\ &= 2P^2 \mathcal{H}^2 \left[\left(\frac{1}{P} - \bar{C}_2 \right) \left(1 - P \left(\frac{1}{P} - C_2 \right) \right) + C_4 P \right] \\ &= \frac{2}{C_1^2 P} \left(C_2 \left(\frac{1}{P} - \bar{C}_2 \right) + C_4 \right) \end{aligned}$$

from which the third identity follows by simple algebraic manipulations. \square

Remark 8. *If we further impose the condition that $C_1 \bar{C}_2$ is real, then the imaginary part of the third identity becomes*

$$\Im(\square_g C_1 P) = \frac{2(|C_2|^2 - C_4)}{|C_1 P|^2} \Im(C_1 P)$$

which will be useful later. In the global case, we can again match the data at spatial infinity to see that $C_4 = |C_2|^2 - 1 = M^2/q^2 - 1$ (the condition relating C_2 and C_4 in Theorem 2 is directly satisfied); the third identity then reads:

$$(q_E + iq_B)\square_g P = -\frac{2}{q^2 P \bar{P}} (M - (q_E - iq_B)\bar{P})$$

An immediate consequence of the above proposition is that $(\nabla C_1 P)^2$ is real. Writing the complex quantity $C_1 P = y + iz$, where y and z are real-valued, we see that this implies

$$\nabla^a y \nabla_a z = 0$$

Furthermore, by Lemma 4, we have that, with the possible exception on sets of co-dimension ≥ 1 , $t^2 \neq 0$. This leads to the useful observation that, with the possible exception on those points, $(\nabla y)^2$ and $(\nabla z)^2$ cannot simultaneously vanish, and in particular $\nabla_a y$ and $\nabla_a z$ are not simultaneously null, and thus rule out the case where the two are aligned. We summarize in the following

Corollary 9. *Letting $C_1 P = y + iz$, we know that on any open set*

- (1) P is not locally constant
- (2) $\nabla_a y$ and $\nabla_a z$ are mutually orthogonal
- (3) $\nabla_a y$ and $\nabla_a z$ cannot be both null
- (4) $\nabla_a y$ and $\nabla_a z$ cannot be parallel

Replacing $C_1 P$ by $y + iz$, and imposing the condition $C_1 \bar{C}_2$ is real, we can also rewrite

$$t^2 = -\frac{C_1 \bar{C}_1 - 2C_1 \bar{C}_2 y}{y^2 + z^2} - |C_2|^2 + C_4$$

Since \mathcal{H}_{ab} is an anti-self-dual two form with non-vanishing norm, it has two distinct principal null directions, which we denote by \underline{l}^a and l^a , with the normalization $g_{ab} \underline{l}^a l^b = -1$. The alignment of \mathcal{H}_{ab} with \mathcal{F}_{ab} (via vanishing of \mathcal{B}_{ab}) allows the following expressions

$$\begin{aligned} \mathcal{H}_{ab} &= \frac{1}{2C_1 P^2} (\underline{l}_a l_b - l_a \underline{l}_b + i\epsilon_{abcd} \underline{l}^c l^d) \\ \mathcal{F}_{ab} &= \frac{\frac{1}{P} - \bar{C}_2}{C_1 P^2} (\underline{l}_a l_b - l_a \underline{l}_b + i\epsilon_{abcd} \underline{l}^c l^d) \end{aligned}$$

By the assumption $\mathcal{Q}_{abcd} = 0$, the principal null directions of \mathcal{H}_{ab} are repeated null directions of the anti-self-dual Weyl tensor, and thus the space-time is algebraically special (Type D). On a local neighborhood, we can take m, \bar{m} complex smooth vector fields to complete the null tetrad $\{m, \bar{m}, \underline{l}, l\}$ (see Appendix A), and in the tetrad (spinor) formalism, the only non-zero Weyl scalar is

$$(11a) \quad \Psi := \Psi_0 = W(\bar{m}, \underline{l}, m, l) = -\frac{1}{C_1^2 P^3} \left(\frac{1}{P} - \bar{C}_2 \right)$$

the only non-zero component of the Maxwell scalars is

$$(11b) \quad \Upsilon := \Upsilon_0 = \mathcal{H}_{ab} l^a \underline{l}^b = \frac{1}{2C_1 P^2}$$

and the only non-zero component of the Ricci scalars is

$$(11c) \quad \Phi := \Phi_0 = T(\underline{l}, l) = T(m, \bar{m}) = \frac{1}{C_1 \bar{C}_1 P^2 \bar{P}^2}$$

Notice the following symmetry relations

$$(12) \quad \bar{\Psi} = \underline{\Psi}, \quad \bar{\underline{\Psi}} = \Psi, \quad \bar{\Upsilon} = -\Upsilon, \quad \bar{\Phi} = \underline{\Phi} = \Phi$$

Now, from

$$2C_1 P^2 \mathcal{H}_{ab} t^a = -C_1 \nabla_b P$$

we can calculate

$$(13a) \quad \nabla_b y = \underline{l}_b l_a t^a - l_b \underline{l}_a t^a \quad (\nabla y)^2 = 2l_a \underline{l}_b t^a t^b$$

$$(13b) \quad \nabla_b z = \epsilon_{bacd} t^a \underline{l}^c l^d \quad (\nabla z)^2 = 2\underline{l}_a l_b t^a t^b + t^2$$

So we need expressions for $g(t, \underline{l}), g(t, l)$. From the fact that $\mathcal{L}_t \mathcal{H} = 0$, we have

$$[t, \underline{l}]_a l_b + \underline{l}_a [t, l]_b - [t, l]_a \underline{l}_b - l_a [t, \underline{l}]_b = 0$$

which we can contract against l and \underline{l} (using the fact that $[t, l]_a l^a = \partial_t l^2 = 0$) to arrive at

$$(14a) \quad [t, \underline{l}]_a = \underline{l}_a [t, l]_b \underline{l}^b = K_t \underline{l}_a$$

$$(14b) \quad [t, l]_a = l_a [t, \underline{l}]_b l^b = -K_t l_a$$

where the function $K_t := [t, l]_b \underline{l}^b$. Now

$$\partial_t (t_b \underline{l}^b) = \mathcal{L}_t (t_b \underline{l}^b) = K_t t_b \underline{l}^b$$

and similarly

$$\partial_t (t_b l^b) = -K_t t_b l^b$$

Lastly, we compute an expression for t by

$$-\frac{\mathcal{H}^{cb} \nabla_b P}{2P^2} = \frac{1}{4} \mathcal{H}^2 t^c = -\frac{t^c}{4C_1^2 P^4}$$

Therefore, by a direct computation

$$(15) \quad t_c = -(l_a t^a) \underline{l}_c - (\underline{l}_a t^a) l_c - \epsilon_{cabd} (\nabla^a z) \underline{l}^b l^d$$

Next is the main lemma of this section

Lemma 10. *Assuming \mathcal{B}_{ab} and \mathcal{Q}_{abcd} vanish, $C_1 \bar{C}_2$ is real, and $|C_2|^2 - C_4 = 1$, we have the norms*

$$(16a) \quad (\nabla z)^2 = \frac{\mathfrak{A} - z^2}{y^2 + z^2}$$

$$(16b) \quad (\nabla y)^2 = \frac{\mathfrak{A} + y^2 + |C_1|^2 - 2C_1 \bar{C}_2 y}{y^2 + z^2}$$

where \mathfrak{A} is a non-negative constant with $z^2 \leq \mathfrak{A}$.

Proof. We will use the tetrad formalism of Klainerman-Ionescu (see Appendix A) extensively in the following computation. By the alignment properties (11) and the symmetry properties (12), the Maxwell equations simplify to

$$\begin{aligned} \underline{D}\Upsilon &= -2\bar{\theta}\Upsilon & D\Upsilon &= -2\theta\Upsilon \\ -\delta\Upsilon &= 2\eta\Upsilon & -\bar{\delta}\Upsilon &= 2\bar{\eta}\Upsilon \end{aligned}$$

from which we arrive at

$$(17) \quad DP = \theta P, \quad \underline{D}P = \bar{\theta}P, \quad \delta P = \eta P, \quad \bar{\delta}P = \bar{\eta}P$$

From the decomposition (13) we then have

$$(18a) \quad \nabla_a y = -\theta C_1 P \underline{l}_a - \bar{\theta} C_1 P l_a$$

$$(18b) \quad i\nabla_a z = \eta C_1 P \bar{m}_a + \bar{\eta} C_1 P m_a$$

Using the fact that y and z are real, taking complex conjugates on the above equations gives us

$$(19) \quad \theta C_1 P = \bar{\theta} \bar{C}_1 \bar{P} \ , \quad \bar{\theta} \bar{C}_1 \bar{P} = \underline{\theta} C_1 P \ , \quad \eta C_1 P = -\underline{\eta} \bar{C}_1 \bar{P}$$

The Bianchi equations become

$$(20a) \quad 0 = \xi(3\Psi + \Phi)$$

$$(20b) \quad 0 = \vartheta(3\Psi - \Phi)$$

$$(20c) \quad -D(\Psi + \frac{1}{2}\Phi) = 3\theta\Psi + \bar{\theta}\Phi$$

$$(20d) \quad \bar{\delta}(\Psi - \frac{1}{2}\Phi) = -3\underline{\eta}\Psi + \bar{\eta}\Phi$$

$$(20e) \quad -\delta\Phi = 2(\eta + \underline{\eta})\Phi$$

$$(20f) \quad D\Phi = -2(\bar{\theta} + \theta)\Phi$$

Because of the triple alignment given by $\mathcal{B}_{ab} = 0$ and $\mathcal{Q}_{abcd} = 0$, the latter four equations contain essentially the same information as the Maxwell equations. We examine the first two in more detail. Consider the equation $3\Psi \pm \Phi = 0$. This implies

$$3\bar{C}_1 \bar{C}_2 \bar{P}^2 - 3\bar{C}_1 \bar{P} \pm C_1 P = 0$$

or

$$\begin{aligned} \frac{3C_1 \bar{C}_2}{C_1 \bar{C}_1} (y^2 - z^2) - (3 \mp 1)y &= 0 \\ \frac{6C_1 \bar{C}_2}{C_1 \bar{C}_1} yz - (3 \pm 1)z &= 0 \end{aligned}$$

Taking derivatives, we have

$$\begin{aligned} (\frac{6C_1 \bar{C}_2}{C_1 \bar{C}_1} y - 3 \pm 1) \nabla y &= \frac{6C_1 \bar{C}_2}{C_1 \bar{C}_1} z \nabla z \\ (\frac{6C_1 \bar{C}_2}{C_1 \bar{C}_1} y - 3 \mp 1) \nabla z &= -\frac{6C_1 \bar{C}_2}{C_1 \bar{C}_1} z \nabla y \end{aligned}$$

By assumption that $C_1 \bar{C}_2$ is real, all the coefficients in the above two equations are real. Suppose the equation $3\Psi \pm \Phi = 0$ is satisfied on an open-set, as ∇y and ∇z cannot be parallel by Corollary 9, we must have then

$$(\frac{6C_1 \bar{C}_2}{C_1 \bar{C}_1} y - 3 \pm 1) \nabla y = \frac{6C_1 \bar{C}_2}{C_1 \bar{C}_1} z \nabla z = 0$$

This implies that y and z are locally constant, which contradicts statement (1) in Corollary 9. Therefore an equation of the form $3\Psi \pm \Phi = 0$ cannot be satisfied on open sets.

Applying to the Bianchi identities (20a,20b), we see that $\xi = \vartheta = \underline{\xi} = \underline{\vartheta} = 0$. The relevant null structure equations, simplified with the above observation, are

$$(21a) \quad (D + \Gamma_{124})\eta = \theta(\underline{\eta} - \eta)$$

$$(21b) \quad -\delta\theta = \zeta\theta + \eta(\theta - \bar{\theta})$$

Define the quantity $A = C_1 \bar{C}_1 P \bar{P} (\nabla z)^2$. Equations (18b) and (19) implies that $(\nabla z)^2 = 2\eta\bar{\eta}C_1\bar{C}_1P\bar{P}$, so

$$\begin{aligned} 0 \leq A &= 2\eta\bar{\eta}C_1^2\bar{C}_1^2P^2\bar{P}^2 \\ &= 2C_1^2\bar{C}_1^2\underline{\eta}\bar{\eta}P^2\bar{P}^2 \\ &= -(y^2 + z^2) - (C_1\bar{C}_1 - 2C_1\bar{C}_2y) - 2\theta\underline{\theta}C_1^2\bar{C}_1^2P^2\bar{P}^2 \end{aligned}$$

where in the last line we used Proposition 7, Corollary 9, Equations (18a) and (19), and the assumption that $|C_2|^2 - C_4 = 1$. By using (21a,21b) we calculate

$$\begin{aligned} D(\eta\bar{\eta}) &= \theta(\underline{\eta} - \eta)\bar{\eta} + \bar{\theta}(\bar{\underline{\eta}} - \bar{\eta})\eta \\ \delta(\theta\underline{\theta}) &= -\eta(\theta - \bar{\theta})\underline{\theta} - \underline{\eta}(\underline{\theta} - \bar{\underline{\theta}})\theta \end{aligned}$$

Thus, with judicious applications of (19)

$$\begin{aligned} DA &= 2C_1^2\bar{C}_1^2[\theta(\underline{\eta} - \eta)\bar{\eta} + \bar{\theta}(\bar{\underline{\eta}} - \bar{\eta})\eta]P^2\bar{P}^2 + 4C_1^2\bar{C}_1^2\eta\bar{\eta}(\theta + \bar{\theta})P^2\bar{P}^2 \\ &= 0 \\ \delta A &= -\delta(z^2) + 2C_1^2\bar{C}_1^2P^2\bar{P}^2[\eta(\theta - \bar{\theta})\underline{\theta} + \underline{\eta}(\underline{\theta} - \bar{\underline{\theta}})\theta] \\ &\quad - 4C_1^2\bar{C}_1^2P^2\bar{P}^2(\eta + \underline{\eta})\theta\underline{\theta} \\ &= -\delta(z^2) \end{aligned}$$

Since $Dz = \underline{D}z = 0$, we have that the function $A + z^2$ is constant. Define $\mathfrak{A} = A + z^2$. The nonnegativity of A guarantees that $z^2 \leq \mathfrak{A}$, and we have

$$(\nabla z)^2 = \frac{A}{C_1\bar{C}_1P\bar{P}} = \frac{\mathfrak{A} - z^2}{y^2 + z^2}$$

and

$$(\nabla y)^2 = (C_1\nabla P)^2 + (\nabla z)^2 = \frac{\mathfrak{A} + y^2 + C_1\bar{C}_1 - 2C_1\bar{C}_2y}{(y^2 + z^2)}$$

as claimed. \square

Remark 11. In the proof above we showed that $\xi = \vartheta = \underline{\xi} = \underline{\vartheta} = 0$, a conclusion that in the vacuum case [Mar99] is easily reached by the Goldberg-Sachs theorem. It is worth noting that in general, the alignment of the principal null directions of the Maxwell form and the Weyl tensor is not enough to justify the vanishing of all four of the involved quantities. Indeed, the Kundt-Thompson theorem [SKM⁺02] only guarantees that $\xi\vartheta = \underline{\xi}\underline{\vartheta} = 0$. In our special case the improvement comes from the fact that we not only have alignment of the principal null directions, but also knowledge of the proportionality factor. This allows us to write down the polynomial expression in P and \bar{P} which we used to eliminate the case where only one of ξ and ϑ vanishes.

In the remainder of this section, we assume that $C_1\bar{C}_2$ is real and $|C_2|^2 - C_4 = 1$ and prove assertions (3) and (4) in Theorem 2. Let us first define two auxillary vector fields. On our space-time, let

$$(22) \quad n^a = (\mathfrak{A} + y^2)t^a + (y^2 + z^2)(t_b l^b \underline{l}^a + t_b \underline{l}^b l^a)$$

Define $\mathcal{M}_{\mathfrak{A}} := \{p \in \mathcal{M} | z^2(p) < \mathfrak{A}\}$. On this open subset we can define

$$(23) \quad b^a = \frac{\nabla^a z}{(\nabla z)^2} = \frac{y^2 + z^2}{\mathfrak{A} - z^2} \nabla^a z$$

We also define the open subsets $\mathcal{M}_l := \{p \in \mathcal{M} | (t_a l^a)(p) \neq 0\}$ and $\mathcal{M}_{\underline{l}} := \{p \in \mathcal{M} | (t_a \underline{l}^a)(p) \neq 0\}$. Now, notice that in our calulations above using the tetrad formalism, we have only specified the “direction” of l, \underline{l} and their lengths relative to each other. We still have considerable freedom left to fix the lapse of one of the two vector fields and still retain the use of our formalism. On $\mathcal{M}_{\underline{l}}$, we can choose the vector field \underline{l} such that $t_a \underline{l}^a = 1$ (similarly for l on \mathcal{M}_l ; the calculations with respect to \mathcal{M}_l are almost identical to that on $\mathcal{M}_{\underline{l}}$, so without loss of generality, we will perform calculations below with respect to $\mathcal{M}_{\underline{l}}$) and the vector field l maintaining $l_a \underline{l}^a = -1$. From (13) and Lemma 10, we have that on $\mathcal{M}_{\underline{l}}$ we can write

$$(24) \quad \nabla_a y = -l_a + \frac{\mathfrak{A} + y^2 + |C_1|^2 - 2C_1\bar{C}_2 y}{2(y^2 + z^2)} \underline{l}_a = -l_a + U \underline{l}_a$$

which implies $l_a t^a = U$, where U is defined on the entirety of \mathcal{M} as

$$(25) \quad U := l_a t^a \underline{l}_b t^b = \frac{1}{2}(\nabla y)^2$$

We consider first a special case when t^a is hypersurface orthogonal.

Proposition 12. *The following are equivalent:*

- (1) z is locally constant on an open subset $\mathcal{U} \subset \mathcal{M}$
- (2) \mathfrak{A} vanishes on \mathcal{M}
- (3) z vanishes on \mathcal{M}

Proof. (2) \implies (3) and (3) \implies (1) follows trivially from Lemma 10. It thus suffices to show (1) \implies (2). Suppose $\nabla z = 0|_{\mathcal{U}}$. We consider the imaginary part of the third identity in Proposition 7 à la Remark 8, which shows that $z = 0|_{\mathcal{U}}$. From Lemma 10 we have $\mathfrak{A} = 0|_{\mathcal{U}}$, but \mathfrak{A} is a universal constant for the manifold, and thus vanishes identically. \square

It is simple to check that $z = 0$ on \mathcal{M} implies $C_1^{-1} \mathcal{H}_{ab} t^a = \nabla_b \frac{1}{C_1 P}$ is real, and so the vanishing of \mathcal{B}_{ab} implies $\mathcal{F}_{ab} t^a = 2(\frac{C_1 \bar{C}_1}{C_1 P} - C_1 \bar{C}_2) C_1^{-1} \mathcal{H}_{ab} t^a$ is purely real, which by Frobenius’ theorem gives that t^a is hypersurface orthogonal.²

Proposition 13. *Assume $\mathfrak{A} = 0$. Then, at any point $p \in \mathcal{M}_{\underline{l}}$ there exists a neighborhood that can be isometrically embedded into the Reissner-Nordström solution.*

This proof closely mirrors that of Proposition 2 in [Mar99].

²As to the question whether t^a can be hypersurface orthogonal without $\nabla z = 0$: in the next part we will consider the case where $\mathfrak{A} \neq 0$ (implying z is nowhere locally constant), and show that in the subset $\mathcal{M}_{\underline{l}} \cap \mathcal{M}_{\mathfrak{A}}$ we have local diffeomorphisms to the Kerr-Newman space-time with non-zero angular momentum, which implies that $\mathfrak{A} = 0$ is characteristic of the Reissner-Nordström metric. Indeed, as we shall see later, the quantity \mathfrak{A} is actually square of the normalized angular momentum of the space-time.

Proof. We use the same tetrad notation as before. Since $z = 0$, we have $C_1 P = y$ is real, and hence (19) implies that $\theta, \underline{\theta}$ are real. Furthermore, $z = 0$ implies via (18b) that $\eta = 0 = \underline{\eta}$. The commutator relations then gives

$$\begin{aligned} [D, \underline{D}] &= -\underline{\omega}D + \omega\underline{D} \\ [\delta, \bar{\delta}] &= \Gamma_{121}\bar{\delta} + \Gamma_{122}\delta \end{aligned}$$

which implies that $\{l, \underline{l}\}$ and $\{m, \bar{m}\}$ are integrable. Thus a sufficiently small neighborhood \mathcal{U} can be foliated by 2 mutually orthogonal families of surfaces. We calculate the induced metric on the surface tangent to $\{m, \bar{m}\}$ using the Gauss equation.

First we calculate the second fundamental form $\chi(X, Y)$ for $X^a = X_1 m^a + X_2 \bar{m}^a$ and $Y^a = Y_1 m^a + Y_2 \bar{m}^a$. By definition $\chi(X, Y)$ is the projection of $\nabla_X Y$ to the normal bundle, so in the tetrad frame, evaluating using the connection coefficients, we have

$$\begin{aligned} \chi(X, Y)^a &= X_1 Y_1 (\Gamma_{131} l^a + \Gamma_{141} \underline{l}^a) + X_1 Y_2 (\Gamma_{231} l^a + \Gamma_{241} \underline{l}^a) \\ &\quad + X_2 Y_1 (\Gamma_{132} l^a + \Gamma_{142} \underline{l}^a) + X_2 Y_2 (\Gamma_{232} l^a + \Gamma_{242} \underline{l}^a) \\ &= X_1 Y_1 (\vartheta l^a + \vartheta \underline{l}^a) + X_1 Y_2 (\bar{\theta} l^a + \bar{\theta} \underline{l}^a) \\ &\quad + X_2 Y_1 (\theta l^a + \theta \underline{l}^a) + X_2 Y_2 (\bar{\vartheta} l^a + \bar{\vartheta} \underline{l}^a) \\ &= -\frac{\nabla^a y}{C_1 P} g(X, Y) = -\frac{\nabla^a y}{y} g(X, Y) \end{aligned}$$

where the last line used the vanishing of ϑ derived in the proof of Lemma 10 and Equation (18a). We recall the Gauss equation

$$R_0(X, Y, Z, W) = R(X, Y, Z, W) - g(\chi(X, W), \chi(Y, Z)) + g(\chi(X, Z), \chi(Y, W))$$

where X, Y, Z, W are spanned by m, \bar{m} . Plugging in the explicit form of the Riemann curvature tensor, we can compute by taking $X = Z = m, Y = W = \bar{m}$ the only component of the curvature tensor for the 2-surface

$$\begin{aligned} R_0(m, \bar{m}, m, \bar{m}) &= -\Psi - \bar{\Psi} - \Phi - \frac{(\nabla y)^2}{y^2} \\ &= \frac{C_1 \bar{C}_1}{y^4} - \frac{2C_1 \bar{C}_2}{y^3} - \frac{(\nabla y)^2}{y^2} = -\frac{1}{y^2} \end{aligned}$$

using Lemma 10 in the last equality. Now, since $\delta y = 0$, we have that the scalar curvature is constant on the 2-surface, and positive, which means that its induced metric is locally the standard metric for S^2 with radius $|y|$. Now, since $\nabla y \neq 0$ on our open set, it is possible to choose a local coördinate system $\{x^0, y, x^2, x^3\}$ compatible with the foliation. Looking at (15) we see that t^a is non-vanishing inside \mathcal{M}_L , and is in fact tangent to the 2-surface formed by $\{l, \underline{l}\}$, so we can take $t = t_x \partial_{x^0}$ for some function t_x . The fact that t^a is Killing gives that $\partial_{x^A} t_x = 0$ for $A = 2, 3$. Recall that we are working in \mathcal{M}_L , and we assumed that $\underline{l}_a t^a = 1$, then we can write, by (24), $\underline{l} = \partial_y + s_x \partial_{x^0}$ for some function s_x . The commutator identity

$$[D, \delta] = -(\Gamma_{124} + \bar{\theta})\delta + \zeta D$$

shows that $\partial_{x^A} s_x = 0$ by considering the decomposition we have for \underline{l} in terms of the coördinate vector fields. Then the Killing relation $[t, \underline{l}] = 0$, together with the above, implies that we can chose a coördinate system $\{u, y, x^2, x^3\}$ with $\partial_u = t$

and $\partial_y = \underline{l}$ that is compatible with the foliation. Lastly, we want to calculate $g_{AB} = g(\partial_{x^A}, \partial_{x^B})$ in this coordinate system. To do so, we use the fact that

$$-l^a = t^a + U \underline{l}^a$$

Then the second fundamental form can be written as

$$\begin{aligned} \chi(X, Y) &= (\nabla_X Y)^\perp \\ &= -(\nabla_X Y)^a (\underline{l}_a l^b + l_a \underline{l}^b) \\ &= -(\nabla_X Y)^a (\underline{l}_a l^b - (t_a + U \underline{l}_a) \underline{l}^b) \end{aligned}$$

Now, when X, Y are tangential fields, since U only depends on y (recall that $z = \mathfrak{A} = 0$), we have that $\nabla_X U = 0$. Furthermore, we use $g(Y, l) = g(Y, t) = 0$ to see

$$\chi(X, Y)^b = l^b Y^a X^c \nabla_c \underline{l}_a - \underline{l}^b Y^a X^c \nabla_c t_a - \underline{l}^b U Y^a X^c \nabla_c \underline{l}_a$$

So we have, using the fact that the second fundamental form is symmetric

$$\begin{aligned} 2\chi(X, Y)^b &= Y^a X^c (l^b - U \underline{l}^b) \mathcal{L}_{\underline{l}} g_{ac} - Y^a X^c \underline{l}^b \mathcal{L}_t g_{ac} \\ &= -Y^a X^c \mathcal{L}_{\underline{l}} g_{ac} \nabla^b y \end{aligned}$$

Taking X and Y to be coordinate vector fields, we conclude that

$$\partial_y g_{AB} = \frac{2}{y} g_{AB}$$

so that $g_{AB} = y^2 g_{AB}^0$ where g_{AB}^0 only depends on x^2, x^3 . Imposing the condition that g_{AB} be the matrix for the standard metric on a sphere of radius $|y|$, we finally conclude that the line element can be written as

$$ds^2 = -(1 - \frac{2C_1 \bar{C}_2 y - |C_1|^2}{y^2}) du^2 + 2dud y + y^2 d\omega_{\mathbb{S}^2}$$

and thus the neighborhood can be embedded into Reissner-Nordström space-time of mass $C_1 \bar{C}_2$ and charge $|C_1|$. \square

Notice that a priori there is no guarantee that $C_1 \bar{C}_2 y > 0$, this is compatible with the fact that we did not specify, for the local version of the theorem, the requirement for asymptotic flatness, and hence are in a case where the mass is not necessarily positive.

Next we consider the general case where t^a is not hypersurface orthogonal. In view of Proposition 12, we can assume that $\mathfrak{A} > 0$ and z not locally constant on any open set. Then it is clear that the set $\mathcal{M}_{\mathfrak{A}}$ is in fact dense in \mathcal{M} : for if there exists an open set on which $z = \mathfrak{A}$, then Proposition 12 implies that $\mathfrak{A} = 0$ identically on \mathcal{M} . Therefore, the set $(\mathcal{M}_{\underline{l}} \cup \mathcal{M}_l) \cap \mathcal{M}_{\mathfrak{A}}$ is non-empty as long as $\mathcal{M}_{\underline{l}} \cup \mathcal{M}_l$ is non-empty; this latter fact can be assured since by assumption (A4) that t^a is timelike at some point $p \in \mathcal{M}$, whereas l^a and \underline{l}^a are non-coïncidental null vectors, so in a neighborhood of p , we must have $l^a t_a \neq 0 \neq \underline{l}^a t_a$. It is on this set that we consider the next proposition.

Proposition 14. *Assuming $\mathfrak{A} > 0$. Let $p \in \mathcal{U} \subset \mathcal{M}_{\underline{l}} \cap \mathcal{M}_{\mathfrak{A}}$ such that t^a, n^a, b^a and \underline{l}^a are well-defined on \mathcal{U} , with normalization $\underline{l}^a t_a = 1$. Then the four vector fields form a holonomic basis, and U can be isometrically embedded into a Kerr-Newman space-time.*

Before giving the proof, we first record the metric for the Kerr-Newman solution in Kerr coördinates

$$(26) \quad ds^2 = - \left(1 - \frac{2Mr - q^2}{r^2 + a^2 \cos^2 \theta} \right) dV^2 + 2drdV + (r^2 + a^2 \cos^2 \theta) d\theta^2 \\ + \frac{[(r^2 + a^2)^2 - (r^2 - 2Mr + a^2 + q^2)a^2 \sin^2 \theta] \sin^2 \theta}{r^2 + a^2 \cos^2 \theta} d\phi^2 \\ - 2a \sin^2 \theta d\phi dr - \frac{2a(2Mr - q^2)}{r^2 + a^2 \cos^2 \theta} \sin^2 \theta dV d\phi$$

Notice that the metric is regular at $r = M \pm \sqrt{M^2 - a^2 - q^2}$ the event and Cauchy horizons.

Proof. We first note that in \mathcal{M}_L , we have the normalization

$$n^a = (y^2 + z^2)(l^a + U\underline{l}^a) + (\mathfrak{A} + y^2)t^a$$

For the proof, it suffices to establish that the commutators between $n^a, b^a, \underline{l}^a, t^a$ vanish and that the vectors are linearly independent (for holonomy), and to calculate their relative inner-products to verify that they define a coördinates equivalent to the Kerr coördinate above.

First we show that the commutators vanish. The cases $[t, \cdot]$ are trivial. Since we fixed $\underline{l}^a t_a = 1$, we have that

$$0 = t^b \nabla_b (\underline{l}_a t^a) = K_t t_b \underline{l}^b = K_t$$

so that $K_t = 0$ and thus $[t, \underline{l}] = [t, l] = 0$. Since y and z are geometric quantities defined from \mathcal{H}_{ab} , and U is a function only of y and z , they are symmetric under the action of t^a , therefore $[t, n] = 0$. Similarly, to evaluate $[t, b]$, it suffices to consider $[t, \nabla z]$. Using (13) we see that ∇z is defined by the volume form, the metric, and the vectors $t^a, \underline{l}^a, l^a$, all of which symmetric under t -action, and thus $[t, b] = 0$. The remaining cases require consideration of the connection coefficients. In view of the normalization condition imposed, $\nabla_a y = -l_a + U\underline{l}_a$, so (18a) implies $\underline{\theta}\bar{C}_1\bar{P} = 1$, $\theta C_1 P = -U$. Recall the null structure equation

$$-\delta\underline{\theta} = -\zeta\underline{\theta} + \underline{\eta}(\underline{\theta} - \bar{\underline{\theta}})$$

Using

$$0 = \delta(\underline{\theta}\bar{C}_1\bar{P}) = (\delta\underline{\theta})\bar{C}_1\bar{P} + \underline{\theta}\eta\bar{C}_1\bar{P}$$

we have

$$\bar{C}_1\bar{P}(\underline{\theta}\eta + \zeta\underline{\theta} - \underline{\eta}\underline{\theta} + \underline{\eta}\bar{\underline{\theta}}) = 0$$

Applications of (19) allows us to replace $+\underline{\eta}\bar{\underline{\theta}}$ by $-\underline{\eta}\underline{\theta}$ in the brackets, and so, since $\underline{\theta}\bar{C}_1\bar{P} = \underline{D}C_1P \neq 0$, we must have $\zeta = \eta$, which considerably simplifies calculations. Next we write

$$b^a = -i \frac{y^2 + z^2}{\mathfrak{A} - z^2} (\eta C_1 P \bar{m}^a - \bar{\eta} \bar{C}_1 \bar{P} m^a) = i \frac{1}{\mathfrak{A} - z^2} (\underline{\eta} C_1 \bar{C}_1^2 P \bar{P}^2 \bar{m}^a - c.c.)$$

by expanding $\nabla^a z$ in tetrad coefficients, and where *c.c.* denotes complex conjugate. Then, since $\underline{D}z = 0$,

$$-i(\mathfrak{A} - z^2)[\underline{l}, b] = \underline{D}(\underline{\eta} C_1 \bar{C}_1 P \bar{P}^2) \bar{m}^a - c.c. + \underline{\eta} C_1 \bar{C}_1^2 P \bar{P}^2 [\underline{D}, \bar{\delta}] - c.c.$$

We consider the commutator relation, simplified appropriately in view of computations above and in the proof of Lemma 10,

$$[\underline{D}, \bar{\delta}] = -(\Gamma_{213} + \underline{\theta})\bar{\delta} = (\Gamma_{123} - \frac{1}{\bar{C}_1 P})\bar{\delta}$$

together with the structure equation

$$(\underline{D} + \Gamma_{123})\underline{\eta} = \underline{\theta}(\eta - \underline{\eta})$$

and the relations in (19) and (17), we get

$$\begin{aligned} \underline{D}(\underline{\eta} C_1 \bar{C}_1^2 P \bar{P}^2) \bar{m}^a + \underline{\eta} C_1 \bar{C}_1^2 P \bar{P}^2 [\underline{D}, \bar{\delta}] \\ = (\underline{D} + \Gamma_{123}) \underline{\eta} C_1 \bar{C}_1^2 P \bar{P}^2 \bar{m}^a - \underline{\eta} |C_1 P|^2 \bar{m}^a + \underline{\eta} \underline{D}(C_1 \bar{C}_1^2 P \bar{P}^2) \bar{m}^a \\ = \underline{\theta}(\eta - \underline{\eta}) C_1 \bar{C}_1^2 P \bar{P}^2 \bar{m}^a - \underline{\eta} |C_1 P|^2 \bar{m}^a + \underline{\eta} (\underline{\theta} \bar{C}_1^3 \bar{P}^3 + 2 \underline{\theta} C_1 \bar{C}_1^2 P \bar{P}^2) \bar{m}^a \\ = 0 \end{aligned}$$

Hence $[\underline{l}, b] = 0$. In a similar fashion, we write

$$n^a = |C_1 P|^2 l^a + \frac{1}{2}(\mathfrak{A} + y^2 + |C_1|^2 - 2C_1 \bar{C}_2 y) \underline{l}^a + (\mathfrak{A} + y^2) t^a$$

From the fact that $b^a \nabla_a y = 0$ and from the known commutator relations, we have

$$[n, b] = [C_1 \bar{C}_1 P \bar{P} l, b] + \frac{1}{2}(\mathfrak{A} + y^2 + |C_1|^2 - 2C_1 \bar{C}_2 y) [\underline{l}, b] + (\mathfrak{A} + y^2) [t, b]$$

of which the second and third terms are already known to vanish. We evaluate $[C_1 \bar{C}_1 P \bar{P} l, b]$ in the same way we evaluated $[\underline{l}, b]$, and a calculation shows that it also vanishes. To evaluate $[\underline{l}, n]$, we need to calculate $[\underline{l}, l]$. To do so we write

$$t^a = -U \underline{l}^a - l^a - \bar{\eta} C_1 P m^a - \eta \bar{C}_1 \bar{P} \bar{m}^a$$

Since $[\underline{l}, t] = 0$, we infer

$$\begin{aligned} [\underline{l}, l] &= -[\underline{l}, U \underline{l} + \bar{\eta} C_1 P m + \eta \bar{C}_1 \bar{P} \bar{m}] \\ &= -\underline{D} U \underline{l} - [\underline{l}, \frac{1}{|C_1 P|^2} \bar{\eta} C_1^2 \bar{C}_1 P^2 \bar{P} m] - c.c. \end{aligned}$$

Notice that in the proof above for $[\underline{l}, b] = 0$ we have demonstrated that $[\underline{l}, \bar{\eta} C_1^2 \bar{C}_1 P^2 \bar{P} m] = 0$, so

$$[\underline{l}, l] = -\underline{D} U \underline{l} + \frac{\underline{D}(|C_1 P|^2)}{|C_1 P|^2} (\bar{\eta} C_1 P m + \eta \bar{C}_1 \bar{P} \bar{m})$$

Direct computation yields

$$\underline{D} U = \frac{y - C_1 \bar{C}_2}{y^2 + z^2} - \frac{2yU}{y^2 + z^2}$$

and

$$\underline{D}(C_1 \bar{C}_1 P \bar{P}) = 2y$$

(recall that we set $\underline{D}y = 1$) so we conclude that

$$[\underline{l}, l] = -\frac{y - C_1 \bar{C}_2}{y^2 + z^2} \underline{l} - \frac{2y}{y^2 + z^2} (l + t)$$

So, using the decomposition for n^a given above

$$\begin{aligned} [\underline{L}, n] &= [\underline{L}, (y^2 + z^2)l + (y^2 + z^2)U\underline{L} + (\mathfrak{A} + y^2)t] \\ &= 2yl + (y - 2C_1\bar{C}_2)\underline{L} + 2yt + (y^2 + z^2)[\underline{L}, l] \\ &= 0 \end{aligned}$$

Having checked the commutators, we now calculate the scalar products between various components. A direct computation from the definition yields

$$\begin{aligned} b^2 &= \frac{y^2 + z^2}{\mathfrak{A} - z^2} & b \cdot n &= 0 & b \cdot \underline{L} &= 0 & b \cdot t &= 0 \\ \underline{L} \cdot n &= \mathfrak{A} - z^2 & \underline{L}^2 &= 0 & \underline{L} \cdot t &= 1 \\ t \cdot n &= \frac{(|C_1|^2 - 2C_1\bar{C}_2y)(z^2 - \mathfrak{A})}{y^2 + z^2} & t^2 &= -1 - \frac{|C_1|^2 - 2C_1\bar{C}_2y}{y^2 + z^2} \end{aligned}$$

and

$$n^2 = (\mathfrak{A} - z^2) \left[\mathfrak{A} + y^2 - \frac{\mathfrak{A} - z^2}{y^2 + z^2} (|C_1|^2 - 2C_1\bar{C}_2y) \right]$$

A simple computation shows that the determinant of the matrix of inner products yields

$$|\det| = (y^2 + z^2)^2 \neq 0$$

and therefore the vector fields are linearly independent. Thus we have shown that they form a holonomic basis.

To construct the local isometry to Kerr-Newman space-time, we define coördinates attached to the holonomic vector fields t, \underline{L}, b, n with the following rescalings. First, since $\mathfrak{A} > 0$, we can define $a > 0$ such that $\mathfrak{A} = a^2$. Then we can define the coördinates r, θ, V, ϕ by

$$\begin{aligned} t &= \partial_V \\ \underline{L} &= \partial_r & y &= r \\ b &= \frac{1}{a \sin \theta} \partial_\theta & z &= a \cos \theta \\ n &= -a \partial_\phi \end{aligned}$$

Notice that we can define θ from z in a way that makes sense since $z^2 \leq \mathfrak{A}$. Applying the change of coördinates to the inner-products above we see that in r, θ, V, ϕ the metric is identical to the one for the Kerr coördinate of Kerr-Newman space-time. Furthermore, we see that n , or ∂_ϕ , defines the corresponding axial Killing vector field. \square

To finish this section, we need to show that the results we obtained in Propositions 13 and 14 can be extended to the manifold \mathcal{M} , rather than restricted to $(\mathcal{M}_{\underline{L}} \cup \mathcal{M}_l)$ in the former and $(\mathcal{M}_{\underline{L}} \cup \mathcal{M}_l) \cap \mathcal{M}_{\mathfrak{A}}$ in the latter. We shall need the following lemma (Lemma 6 in [Mar99]; the lemma and its proof can be carried over to our case essentially without change, we reproduce them here for completeness)

Lemma 15. *The vector field n^a is a Killing vector field on the entirety of \mathcal{M} . The set $\mathcal{M} \setminus \mathcal{M}_{\mathfrak{A}} = \{n^a = 0\}$. Furthermore,*

- *If $\mathfrak{A} = 0$, then $\mathcal{M} \setminus (\mathcal{M}_{\underline{L}} \cup \mathcal{M}_l) = \{t^a = 0\}$*

- If $0 < \mathfrak{A} \leq (C_1 \bar{C}_2)^2 - |C_1|^2$, then $\mathcal{M} \setminus (\mathcal{M}_l \cup \mathcal{M}_l) = \{ \text{either } n^a - y_+ t^a = 0 \text{ or } n^a - y_- t^a = 0 \}$ where

$$y_{\pm} = 2(C_1 \bar{C}_2)^2 - |C_1|^2 \pm 2C_1 \bar{C}_2 \sqrt{(C_1 \bar{C}_2)^2 - |C_1|^2 - \mathfrak{A}}$$

- If $\mathfrak{A} > (C_1 \bar{C}_2)^2 - |C_1|^2$, then $\mathcal{M} \setminus (\mathcal{M}_l \cup \mathcal{M}_l) = \emptyset$

Proof. First consider the case $\mathfrak{A} = 0$. By Proposition 12, we have $z = 0$. So the definition (22) and (15) show that n^a vanishes identically. Furthermore, since $\mathcal{M}_{\mathfrak{A}} = \emptyset$ in this case, we have that n^a is a (trivial) Killing vector field on \mathcal{M} vanishing on $\mathcal{M} \setminus \mathcal{M}_{\mathfrak{A}}$. It is also clear from (15) that $t^a = 0 \iff t_a l^a = t_a \bar{l}^a = 0$ in this case, proving the first bullet point.

Now let $\mathfrak{A} > 0$. Then Proposition 14 shows that n^a is Killing on $(\mathcal{M}_l \cup \mathcal{M}_l) \cap \mathcal{M}_{\mathfrak{A}}$, and does not coincide with t^a . Since $\mathcal{M}_{\mathfrak{A}}$ is dense in \mathcal{M} (see paragraph immediately before Proposition 14), we have that n^a is Killing on $\overline{\mathcal{M}_l \cup \mathcal{M}_l}$ (the overline denotes set closure). We wish to show that $\overline{\mathcal{M}_l \cup \mathcal{M}_l} = \mathcal{M}$. Suppose not, then the open set $\mathcal{U} = \mathcal{M} \setminus \overline{\mathcal{M}_l \cup \mathcal{M}_l}$ is non-empty. In \mathcal{U} , $t_a l^a = t_a \bar{l}^a = 0$, so by (13), $\nabla^a y = 0$ in \mathcal{U} . Taking the real part of the third identity in Proposition 7, we must have $y = C_1 \bar{C}_2$ in \mathcal{U} , which by Lemma 10 implies $\mathfrak{A} = (C_1 \bar{C}_2)^2 - |C_1|^2$. Consider the vectorfield defined on all of \mathcal{M} given by $n^a - (\mathfrak{A} + y^2)t^a = n^a - [2(C_1 \bar{C}_2)^2 - |C_1|^2]t^a$. As it is a constant coefficient linear combination of non-vanishing independent Killing vector fields on $\overline{\mathcal{M}_l \cup \mathcal{M}_l}$, it is also a non-vanishing Killing vector field. However, on \mathcal{U} , the vector field vanishes by construction. So we have Killing vector field on \mathcal{M} that is not identically 0, yet vanishes on a non-empty open set, which is impossible (see Appendix C.3 in [Wal84]). Therefore n^a is a Killing vector field everywhere on \mathcal{M} . Now, outside of $\mathcal{M}_{\mathfrak{A}}$, we have that $z^2 = \mathfrak{A}$ reaches a local maximum, so $\nabla_a z$ must vanish. Therefore from (22) and (15) we conclude that n^a vanishes outside $\mathcal{M}_{\mathfrak{A}}$ also, proving the second statement in the lemma.

For the second and third bullet points, consider the function $U = \frac{1}{2}(\nabla y)^2$. By definition it vanishes outside $\mathcal{M}_l \cup \mathcal{M}_l$. Using Lemma 10 we see that

$$\mathfrak{A} + y^2 + |C_1|^2 - 2C_1 \bar{C}_2 y = 0$$

outside $\mathcal{M}_l \cup \mathcal{M}_l$. The two bullet points are clear in view of the quadratic formula and (22). \square

Now we can complete the main theorem in the same way as [Mar99].

Proof of the Main Theorem. In view of Propositions 13 and 14, we only need to show that the isometry thus defined extends to $\mathcal{M} \setminus (\mathcal{M}_l \cup \mathcal{M}_l)$ in the case of Reissner-Nordström and $\mathcal{M} \setminus [(\mathcal{M}_l \cup \mathcal{M}_l) \cap \mathcal{M}_{\mathfrak{A}}]$ in the case of Kerr-Newman. Lemma 15 shows that those points we are interested in are fixed points of Killing vector fields, and hence are either isolated points or smooth, two-dimensional, totally geodesic surfaces. Their complement, therefore, are connected and dense, with local isometry into the Kerr-Newman family. Therefore a sufficiently small neighborhood of one of these fixed-points will have a dense and connected subset isometric to a patch of Kerr-Newman, whence we can extend to those fixed-points by continuity. \square

4. PROOF OF THE MAIN GLOBAL RESULT

To show Corollary 3, it suffices to demonstrate that the global assumption (G) leads to the local assumption (L).

By asymptotic flatness and the imposed decay rate (the assumption that the mass and charge at infinity are non-zero), we can assume that there is a simply connected region $\mathcal{M}_{\mathcal{H}}$ near spatial infinity such that $\mathcal{H}^2 \neq 0$. It thus suffices to show that $\mathcal{M}_{\mathcal{H}} = \mathcal{M}$. Suppose not, then the former is a proper subset of the latter. Let $p_0 \in \mathcal{M}$ be a point on $\partial\mathcal{M}_{\mathcal{H}}$. We see that Theorem 2 applies to $\mathcal{M}_{\mathcal{H}}$, with C_1 taken to be $q_E + iq_B$ and $C_3 = M/(q_E - iq_B)$. In particular, the first equation in Proposition 7 shows that, by continuity, $t^2 = -1$ at p_0 . Let δ be a small neighborhood of p_0 such that t^a is everywhere time-like in δ with $t^2 < -\frac{1}{4}$, then the metric g induces a uniform Riemannian metric on the bundle of orthogonal subspaces to t^a , i.e. $\cup_{p \in \delta} \{v \in T_p\mathcal{M} | g(v, t) = 0\}$. Now, consider a curve $\gamma : (s_0, 1] \rightarrow \delta$ such that $\gamma(s) \in \mathcal{M}_{\mathcal{H}}$ for $s < 1$, $\gamma(1) = p_0$, and $\frac{d}{ds}\gamma(s)$ has norm 1 and is orthogonal to t . Consider the function $(q_E + iq_B)P \circ \gamma$. By assumption, $|(q_E + iq_B)P \circ \gamma| \nearrow \infty$ as $s \nearrow 1$. Since Lemma 10 guarantees that z is bounded in $\mathcal{M}_{\mathcal{H}}$, and hence by continuity, at p_0 , we must have that y blows up as we approach p_0 along γ . However,

$$\left| \frac{d}{ds}(y \circ \gamma) \right| = \left| \nabla_{\frac{d}{ds}\gamma} y \right| \leq C \sqrt{|\nabla_a y \nabla^a y|} < C' < \infty$$

where the constant C comes from the uniform control on g acting as a Riemannian metric on the orthogonal subspace to t^a (note that $t^a \nabla_a y = 0$ since y is a quantity derivable from quantities that are invariant under the t -action), and C' arises because by Lemma 10, $\nabla_a y \nabla^a y$ is bounded for all $|y| > 2M$, which we can guarantee for s sufficiently close to 1. So we have a contradiction: $y \circ \gamma$ blows up in finite time while its derivative stays bounded. Therefore $\mathcal{M}_{\mathcal{H}} = \mathcal{M}$.

APPENDIX A. TETRAD FORMALISMS

The null tetrad formalism of Newman and Penrose is used extensively in the calculations above, albeit with slightly different notational conventions. In the following, a dictionary is given between the standard Newman-Penrose variables (see, e.g. Chapter 7 in [SKM⁺02]) and the null-structure variables of Ionescu and Klainerman [IK07a] which is used in this paper.

Following Ionescu and Klainerman [IK07a], we consider a space-time with a natural choice of a *null pair* $\{\underline{l}, l\}$. Recall that the complex valued vector field m is said to be *compatible* with the null pair if

$$g(l, m) = g(\underline{l}, m) = g(m, m) = 0, \quad g(m, \bar{m}) = 1$$

where \bar{m} is the complex conjugate of m . Given a null pair, for any point $p \in \mathcal{M}$, such a compatible vector field always exist on a sufficiently small neighborhood of p . We say that the vector fields $\{m, \bar{m}, \underline{l}, l\}$ form a *null tetrad* if, in addition, they have positive orientation $\epsilon_{abcd} m^a \bar{m}^b \underline{l}^c l^d = i$ (we can always swap m and \bar{m} by the obvious transformation to satisfy this condition).

The scalar functions corresponding to the connection coefficients of the null tetrad are defined, with translation to the Newman-Penrose formalism, in Table 1. The Γ -notation is defined by

$$\Gamma_{\alpha\beta\gamma} = g(\nabla_{e_\gamma} e_\beta, e_\alpha)$$

	Γ -notation	Newman-Penrose	Ionescu-Klainerman
$g(\nabla_{\bar{m}}l, m)$	Γ_{142}	$-\rho$	θ
$g(\nabla_{\bar{m}}\underline{l}, m)$	Γ_{132}	$\bar{\mu}$	$\underline{\theta}$
$g(\nabla_m l, m)$	Γ_{141}	$-\sigma$	ϑ
$g(\nabla_m \underline{l}, m)$	Γ_{131}	$\bar{\lambda}$	$\underline{\vartheta}$
$g(\nabla_l l, m)$	Γ_{144}	$-\kappa$	ξ
$g(\nabla_l \underline{l}, m)$	Γ_{133}	$\bar{\nu}$	$\underline{\xi}$
$g(\nabla_l l, m)$	Γ_{143}	$-\tau$	η
$g(\nabla_l \underline{l}, m)$	Γ_{134}	$\bar{\pi}$	$\underline{\eta}$
$g(\nabla_l l, \underline{l})$	Γ_{344}	$-2\epsilon + \Gamma_{214}$	ω
$g(\nabla_l \underline{l}, \underline{l})$	Γ_{433}	$2\gamma + \Gamma_{123}$	$\underline{\omega}$
$g(\nabla_m l, \underline{l})$	Γ_{341}	$-2\beta + \Gamma_{211}$	$\zeta = -\bar{\zeta}$

TABLE 1. Dictionary of Ricci rotation coefficients vs. Newman-Penrose spin coefficients vs. Ionescu-Klainerman connection coefficients

where for $e_1 = m$, $e_2 = \bar{m}$, $e_3 = \underline{l}$, and $e_4 = l$. It is clear that $\Gamma_{(\alpha\beta)\gamma} = 0$, i.e. it is antisymmetric in the first two indices. Two natural³ operations are then defined: the under-bar (e.g. $\theta \leftrightarrow \underline{\theta}$) corresponds to swapping the indices $3 \leftrightarrow 4$ (e.g. $\Gamma_{142} \leftrightarrow \Gamma_{132}$), and complex conjugation (e.g. $\theta \leftrightarrow \bar{\theta}$) corresponds to swapping the numeric indices $1 \leftrightarrow 2$ (e.g. $\Gamma_{142} \leftrightarrow \Gamma_{241}$). We note that $\theta, \underline{\theta}, \vartheta, \underline{\vartheta}, \xi, \underline{\xi}, \eta, \underline{\eta}, \zeta$ are complex-valued, while ω and $\underline{\omega}$ are real-valued; thus the connection-coefficients defined above, along with complex-conjugation, defines 20 out of the 24 rotation coefficients: the only ones not given a “name” are $\Gamma_{121}, \Gamma_{122}, \Gamma_{123}, \Gamma_{124}$, among which the first two are related by complex-conjugation, and the latter-two by under-bar.

The directional derivative operators are given by:

$$D = l^a \nabla_a, \underline{D} = \underline{l}^a \nabla_a, \delta = m^a \nabla_a, \bar{\delta} = \bar{m}^a \nabla_a$$

(their respective symbols in Newman-Penrose notation are $D, \Delta, \delta, \bar{\delta}$).

The spinor components of the Riemann curvature tensor can be given in terms of the following: let W_{abcd} be the Weyl curvature tensor, S_{ab} be the traceless Ricci

³Buyers beware: the operations are only natural in so much as those geometric statements that are agnostic to orientation of the frame vectors. Indeed, both the under-bar and complex conjugation changes the sign of the Levi-Civita symbol; while for the complex conjugation it is of less consequence (since the complex conjugate of $-i$ is i , the sign difference is most naturally absorbed), for the under-bar operation one needs to take care in application to ascertain that sign-changes due to, say, the Hodge star operator is not present in the equation under consideration. In particular, generally coördinate independent geometric statements (such as the relations to be developed in this section) will be compatible with consistent application of the under-bar operations, while statements dependent on a particular choice of foliation or frame will usually need to be evaluated on a case-by-case basis.

tensor, and R be the scalar curvature, we can write

$$\begin{aligned}
(27a) \quad & \Psi_2 = W(l, m, l, m) \\
(27b) \quad & \bar{\Psi}_{-2} = \underline{\Psi}_2 = W(\underline{l}, m, \underline{l}, m) \\
(27c) \quad & \Psi_1 = W(m, l, \underline{l}, l) \\
(27d) \quad & \bar{\Psi}_{-1} = \underline{\Psi}_1 = W(m, \underline{l}, l, \underline{l}) \\
(27e) \quad & \Psi_0 = W(\bar{m}, \underline{l}, m, l) \\
(27f) \quad & \Phi_{11} = S(l, l) \\
(27g) \quad & \underline{\Phi}_{11} = S(\underline{l}, \underline{l}) \\
(27h) \quad & \Phi_{01} = S(m, l) \\
(27i) \quad & \underline{\Phi}_{01} = S(m, \underline{l}) \\
(27j) \quad & \Phi_{00} = S(m, m) \\
(27k) \quad & \Phi_0 = \frac{1}{2}[S(l, \underline{l}) + S(m, \bar{m})]
\end{aligned}$$

Notice that the quantities Ψ_A , $A \in \{-2, -1, 0, 1, 2\}$ are automatically anti-self-dual: replacing $W_{abcd} \leftrightarrow *W_{abcd}$ we have $\Psi_A(*W) = (-i)\Psi_A(W)$, this follows from the orthogonality properties of the null tetrad, as well as the orientation requirement $\epsilon(m, \bar{m}, \underline{l}, l) = i$. Using this notation, we can write the *null structure equations*, which are equivalent to the Newman-Penrose equations. We derive them from the definition of the Riemann curvature tensor:

$$R_{\alpha\beta\mu\nu} = e_\mu(\Gamma_{\alpha\beta\nu}) - e_\nu(\Gamma_{\alpha\beta\mu}) + \Gamma_{\beta\nu}^\rho \Gamma_{\alpha\rho\mu} - \Gamma_{\beta\mu}^\rho \Gamma_{\alpha\rho\nu} + (\Gamma_{\mu\nu}^\rho - \Gamma_{\nu\mu}^\rho)\Gamma_{\alpha\beta\rho}$$

and that

$$R_{\alpha\beta\mu\nu} = W_{\alpha\beta\mu\nu} + \frac{1}{2}(S_{\alpha\mu}g_{\beta\nu} + S_{\beta\nu}g_{\alpha\mu} - S_{\alpha\nu}g_{\beta\mu} - S_{\beta\mu}g_{\alpha\nu}) + \frac{1}{12}R(g_{\alpha\mu}g_{\beta\nu} - g_{\beta\mu}g_{\alpha\nu})$$

So from $R_{1441} = W_{1441} = -\Psi_2$ we get

$$(28a) \quad (D + 2\Gamma_{124})\vartheta - (\delta + \Gamma_{121})\xi = \xi(2\zeta + \eta + \underline{\eta}) - \vartheta(\omega + \theta + \bar{\theta}) - \Psi_2$$

by taking under-bar of the whole expression, we get for a similar expression for $R_{1331} = -\underline{\Psi}_2$ (in the interest of space, we omit the obvious changes of variables here). For $R_{1442} = -\frac{1}{2}S_{44}$ (and analogously $R_{1332} = -\frac{1}{2}S_{33}$) we have

$$(28b) \quad D\theta - (\bar{\delta} + \Gamma_{122})\xi = -\theta^2 - \omega\theta - \vartheta\bar{\vartheta} + \bar{\xi}\eta + \xi(2\bar{\zeta} + \bar{\eta}) - \frac{1}{2}\Phi_{11}$$

From $R_{1443} = -\Psi_1 - \frac{1}{2}S_{14}$

$$(28c) \quad (D + \Gamma_{124})\eta - (\underline{D} + \Gamma_{123})\xi = -2\underline{\omega}\xi + \theta(\underline{\eta} - \eta) + \vartheta(\bar{\eta} - \bar{\eta}) - \Psi_1 - \frac{1}{2}\Phi_{01}$$

From $R_{1431} = \frac{1}{2}S_{11}$ we get

$$(28d) \quad (\underline{D} + 2\Gamma_{123})\vartheta - (\delta + \Gamma_{121})\eta = \eta^2 + \xi\underline{\xi} - \theta\underline{\vartheta} + \vartheta(\underline{\omega} - \bar{\theta}) + \frac{1}{2}\Phi_{00}$$

From $R_{1432} = -\Psi_0 + \frac{1}{12}R$ we have

$$(28e) \quad \underline{D}\theta - (\bar{\delta} + \Gamma_{122})\eta = \xi\underline{\bar{\xi}} + \eta\bar{\eta} - \vartheta\underline{\bar{\vartheta}} + \theta(\underline{\omega} - \underline{\theta}) - \Psi_0 + \frac{R}{12}$$

From $R_{1421} = -\Psi_1 + \frac{1}{2}S_{41}$ we have

$$(28f) \quad (\bar{\delta} + 2\Gamma_{122})\vartheta - \delta\theta = \zeta\theta - \bar{\zeta}\vartheta + \eta(\theta - \bar{\theta}) + \xi(\underline{\theta} - \bar{\underline{\theta}}) - \Psi_1 + \frac{1}{2}\Phi_{01}$$

Using $R_{3441} = -\Psi_1 - \frac{1}{2}S_{41}$ we get

$$(28g) \quad (D + \Gamma_{124})\zeta - \delta\omega = \omega(\zeta + \underline{\eta}) + \bar{\theta}(\underline{\eta} - \zeta) + \vartheta(\bar{\eta} - \bar{\zeta}) - \xi(\bar{\underline{\theta}} + \underline{\omega}) - \bar{\xi}\underline{\vartheta} - \Psi_1 - \frac{1}{2}\Phi_{01}$$

From $R_{3443} = \Psi_0 + \bar{\Psi}_0 - S_{34} + \frac{R}{12}$ we get

$$(28h) \quad D\underline{\omega} + \underline{D}\omega = \bar{\xi}\underline{\xi} + \xi\underline{\bar{\xi}} - \bar{\eta}\underline{\eta} - \eta\underline{\bar{\eta}} + \zeta(\bar{\eta} - \underline{\eta}) + \bar{\zeta}(\eta - \underline{\eta}) - (\Psi_0 + \bar{\Psi}_0) + \Phi_0 - \frac{R}{12}$$

and lastly from $R_{3421} = \Psi_0 - \bar{\Psi}_0$ we have

$$(28i) \quad (\delta - \Gamma_{121})\bar{\zeta} - (\bar{\delta} + \Gamma_{122})\zeta = (\bar{\vartheta}\underline{\vartheta} - \vartheta\underline{\bar{\vartheta}}) + (\theta\underline{\bar{\theta}} - \bar{\theta}\underline{\theta}) + \underline{\omega}(\theta - \bar{\theta}) - \omega(\underline{\theta} - \bar{\underline{\theta}}) - (\Psi_0 - \bar{\Psi}_0)$$

In this formalism, we can also write the Maxwell equations: let

$$(29a) \quad \Upsilon_0 = \frac{1}{2}(H(l, \underline{l}) + H(\bar{m}, m)) = \mathcal{H}_{ab}l^a\underline{l}^b$$

$$(29b) \quad \Upsilon_1 = H(l, m) = \mathcal{H}_{ab}l^am^b$$

$$(29c) \quad \tilde{\Upsilon}_{-1} = \underline{\Upsilon}_1 = H(m, \underline{l}) = \tilde{\mathcal{H}}_{ab}m^a\underline{l}^b$$

be the spinor components of the Maxwell two-form H_{ab} . Maxwell's equations becomes

$$(30a) \quad \underline{D}\Upsilon_0 - (\delta - \Gamma_{121})\Upsilon_{-1} = \bar{\xi}\Upsilon_1 - 2\bar{\underline{\theta}}\Upsilon_0 - (\zeta - \eta)\Upsilon_{-1}$$

$$(30b) \quad (\underline{D} + \Gamma_{123})\Upsilon_1 - \delta\Upsilon_0 = (\underline{\omega} - \bar{\underline{\theta}})\Upsilon_1 + 2\eta\Upsilon_0 - \vartheta\Upsilon_{-1}$$

and their under-bar counterparts.

We also need the Bianchi identities

$$\nabla_{[e}R_{ab]cd} = 0$$

Note that this implies

$$\nabla^e W_{ebcd} = \nabla_{[c}S_{d]b} - \frac{1}{12}g_{b[c}\nabla_{d]}R =: J_{bcd}$$

which gives

$$\nabla_{[e}W_{ab]cd} = \frac{1}{6}\epsilon_{seab}J^{srt}\epsilon_{rtcd}$$

using the orientation condition $\epsilon(m, \bar{m}, \underline{L}, l) = i$ we calculate

$$\begin{aligned}
(31a) \quad & (\bar{\delta} + 2\Gamma_{122})\Psi_2 - (D + \Gamma_{124})\Psi_1 + \frac{1}{2}\delta\Phi_{11} - \frac{1}{2}(D + \Gamma_{124})\Phi_{01} \\
& = -(2\bar{\zeta} + \bar{\eta})\Psi_2 + (4\theta + \omega)\Psi_1 + 3\xi\Psi_0 \\
& \quad - (\bar{\theta} + \frac{1}{2}\omega)\Phi_{01} - \vartheta\bar{\Phi}_{01} + (\zeta + \frac{1}{2}\underline{\eta})\Phi_{11} + \xi\Phi_0 + \frac{1}{2}\bar{\xi}\bar{\Phi}_{00} \\
(31b) \quad & (\underline{D} + 2\Gamma_{123})\Psi_2 - (\delta + \Gamma_{121})\Psi_1 + \frac{1}{2}(D + 2\Gamma_{124})\Phi_{00} - \frac{1}{2}(\delta + \Gamma_{121})\Phi_{01} \\
& = (2\underline{\omega} - \bar{\theta})\Psi_2 + (\zeta + 4\eta)\Psi_1 + 3\vartheta\Psi_0 \\
& \quad - \frac{1}{2}\bar{\theta}\Phi_{00} - \vartheta\Phi_0 - \frac{1}{2}\underline{\vartheta}\Phi_{11} + \xi\underline{\Phi}_{01} + (\frac{1}{2}\zeta + \underline{\eta})\Phi_{01} \\
(31c) \quad & -(\bar{\delta} + \Gamma_{122})\Psi_1 - D\Psi_0 - \frac{1}{2}D\Phi_0 + \frac{1}{2}(\delta - \Gamma_{121})\bar{\Phi}_{01} - \frac{1}{24}DR \\
& = -\bar{\vartheta}\Psi_2 + (2\bar{\eta} + \bar{\zeta})\Psi_1 + 3\theta\Psi_0 + 2\xi\bar{\Psi}_1 \\
& \quad - \frac{1}{2}(\zeta + \underline{\eta})\bar{\Phi}_{01} + \bar{\theta}\Phi_0 + \frac{1}{2}\bar{\theta}\Phi_{11} + \frac{1}{2}\vartheta\bar{\Phi}_{00} \\
& \quad - \frac{1}{2}\bar{\xi}\bar{\Phi}_{01} - \frac{1}{2}\bar{\eta}\Phi_{01} - \frac{1}{2}\xi\bar{\Phi}_{01} \\
(31d) \quad & (D + \Gamma_{124})\underline{\Psi}_1 + \delta\bar{\Psi}_0 + \frac{1}{2}(\underline{D} + \Gamma_{123})\Phi_{01} - \frac{1}{2}\delta\Phi_0 + \frac{1}{24}\delta R \\
& = -2\underline{\vartheta}\bar{\Psi}_1 - 3\bar{\eta}\bar{\Psi}_0 + (\omega - 2\bar{\theta})\underline{\Psi}_1 + \bar{\xi}\underline{\Psi}_2 \\
& \quad + \frac{1}{2}(\underline{\omega} - \bar{\theta})\Phi_{01} - \frac{1}{2}\bar{\theta}\Phi_{01} - \frac{1}{2}\vartheta\underline{\Phi}_{01} - \frac{1}{2}\vartheta\bar{\Phi}_{01} \\
& \quad + \frac{1}{2}\bar{\eta}\Phi_{00} + \eta\Phi_0
\end{aligned}$$

In addition, we can also take the trace of the Bianchi identities, which gives

$$0 = \nabla^e W_{ebc}{}^b = J_{bc}{}^b$$

and evaluates to

$$\begin{aligned}
(31e) \quad & -\delta\Phi_0 - (\bar{\delta} + 2\Gamma_{122})\Phi_{00} + (\underline{D} + \Gamma_{123})\Phi_{01} + (D + \Gamma_{124})\underline{\Phi}_{01} + \frac{1}{4}\delta R \\
& = (\bar{\eta} + \bar{\eta})\Phi_{00} + 2(\eta + \underline{\eta})\Phi_0 + (\omega - 2\theta - \bar{\theta})\underline{\Phi}_{01} + (\underline{\omega} - 2\underline{\theta} - \bar{\theta})\Phi_{01} \\
& \quad - \vartheta\underline{\Phi}_{01} - \underline{\vartheta}\bar{\Phi}_{01} + \xi\underline{\Phi}_{11} + \underline{\xi}\Phi_{11} \\
(31f) \quad & D\Phi_0 + \underline{D}\Phi_{11} - (\delta - \Gamma_{121})\bar{\Phi}_{01} - (\bar{\delta} + \Gamma_{122})\Phi_{01} + \frac{1}{4}DR \\
& = -\bar{\vartheta}\Phi_{00} - 2(\bar{\theta} + \theta)\Phi_0 + \bar{\xi}\bar{\Phi}_{01} + (\bar{\zeta} + 2\bar{\eta} + \bar{\eta})\Phi_{01} - \vartheta\bar{\Phi}_{00} \\
& \quad + \xi\bar{\Phi}_{01} + (\zeta + 2\eta + \underline{\eta})\bar{\Phi}_{01} + (2\underline{\omega} - \underline{\theta} - \bar{\theta})\Phi_{11}
\end{aligned}$$

A simple identification using Table 1 and the definitions for various spinor components of the Riemann and traceless Ricci tensors shows that one can recover all of the Bianchi identities in Newman-Penrose formalism from the above six equations through the action of complex-conjugation and under-barring.

Lastly, to complete the formalism, we record the commutator relations

$$\begin{aligned}
 (32a) \quad [D, \underline{D}] &= (\underline{\eta} - \eta)\bar{\delta} + (\bar{\eta} - \bar{\eta})\delta - \underline{\omega}D + \omega\underline{D} \\
 (32b) \quad [D, \delta] &= -\vartheta\bar{\delta} - (\Gamma_{124} + \bar{\theta})\delta + (\underline{\eta} + \zeta)D + \xi\underline{D} \\
 (32c) \quad [\delta, \bar{\delta}] &= \Gamma_{121}\bar{\delta} + \Gamma_{122}\delta + (\bar{\theta} - \underline{\theta})D + (\bar{\theta} - \theta)\underline{D}
 \end{aligned}$$

REFERENCES

- [BCJM04] Donato Bini, Christian Cherubini, Robert T. Jantzen, and Giovanni Miniutti, *The Simon and Simon-Mars tensors for stationary Einstein-Maxwell fields*, Classical and Quantum Gravity **21** (2004), 1987–1998.
- [Bun83] Gary Bunting, *Proof of the uniqueness conjecture for black holes*, Ph.D. thesis, University of New England, Australia, 1983.
- [Car73] Brandon Carter, *Black hole equilibrium states*, Gordon and Breach, 1973.
- [CK93] Demetrios Christodoulou and Sergiu Klainerman, *The global nonlinear stability of the Minkowski space*, Princeton University Press, 1993.
- [DKM84] R. Debever, N. Kamran, and R. G. McLenaghan, *Exhaustive integration and a single expression for the general solution of the type D vacuum and electrovac field equations with cosmological constant for a nonsingular aligned Maxwell field*, Journal of Mathematical Physics **25** (1984), no. 6, 1955–1972.
- [IK07a] Alexandru D. Ionescu and Sergiu Klainerman, *On the uniqueness of smooth, stationary black holes in vacuum*, preprint arXiv:0711.0040v1 [gr-qc] (2007).
- [IK07b] ———, *Uniqueness results for ill posed characteristic problems in curved space-times*, preprint arXiv:0711.0042v1 [gr-qc] (2007).
- [Isr67] Werner Israel, *Event horizons in static vacuum space-times*, Physical review **164** (1967), no. 5, 1776–1779.
- [Isr68] ———, *Event horizons in static electrovac space-times*, Communications in mathematical physics **8** (1968), 245–260.
- [JR05] Nils Voje Johansen and Finn Ravndal, *On the discovery of Birkhoff’s theorem*, preprint arXiv:physics/0508163v2 [physics.hist-ph] (2005).
- [Ker63] Roy P. Kerr, *Gravitational field of a spinning mass as an example of algebraically special metrics*, Physical Review Letters **11** (1963), no. 5, 237–238.
- [Mar99] Marc Mars, *A spacetime characterization of the Kerr metric*, Classical and Quantum Gravity **16** (1999), 2507–2523.
- [Mar00] ———, *Uniqueness properties of the Kerr metric*, Classical and Quantum Gravity **17** (2000), 3353–3373.
- [Maz82] Pawel O. Mazur, *Proof of uniqueness of the Kerr-Newman black hole solution*, Journal of Physics A **15** (1982), 3173–3180.
- [NCC⁺65] Ezra T. Newman, E. Couch, K. Chinnapared, A. Exton, A. Prakash, and R. Torrence, *Metric of a rotating, charged mass*, Journal of Mathematical Physics **6** (1965), no. 6, 918–919.
- [Rob75] D. C. Robinson, *Uniqueness of the Kerr black hole*, Physical Review Letters **34** (1975), no. 14, 905–906.
- [Sim84a] Walter Simon, *Characterizations of the Kerr metric*, General Relativity and Gravitation **16** (1984), no. 5, 465–476.
- [Sim84b] ———, *The multiple expansion of stationary Einstein-Maxwell fields*, Journal of Mathematical Physics **25** (1984), no. 4, 1035–1038.
- [SKM⁺02] Hans Stephani, Dietrich Kramer, Malcolm MacCallum, Cornelius Hoenselaers, and Eduard Herit, *Exact solutions of Einstein’s field equations*, second ed., Cambridge University Press, 2002.
- [Wal84] Robert M. Wald, *General relativity*, University of Chicago Press, 1984.

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